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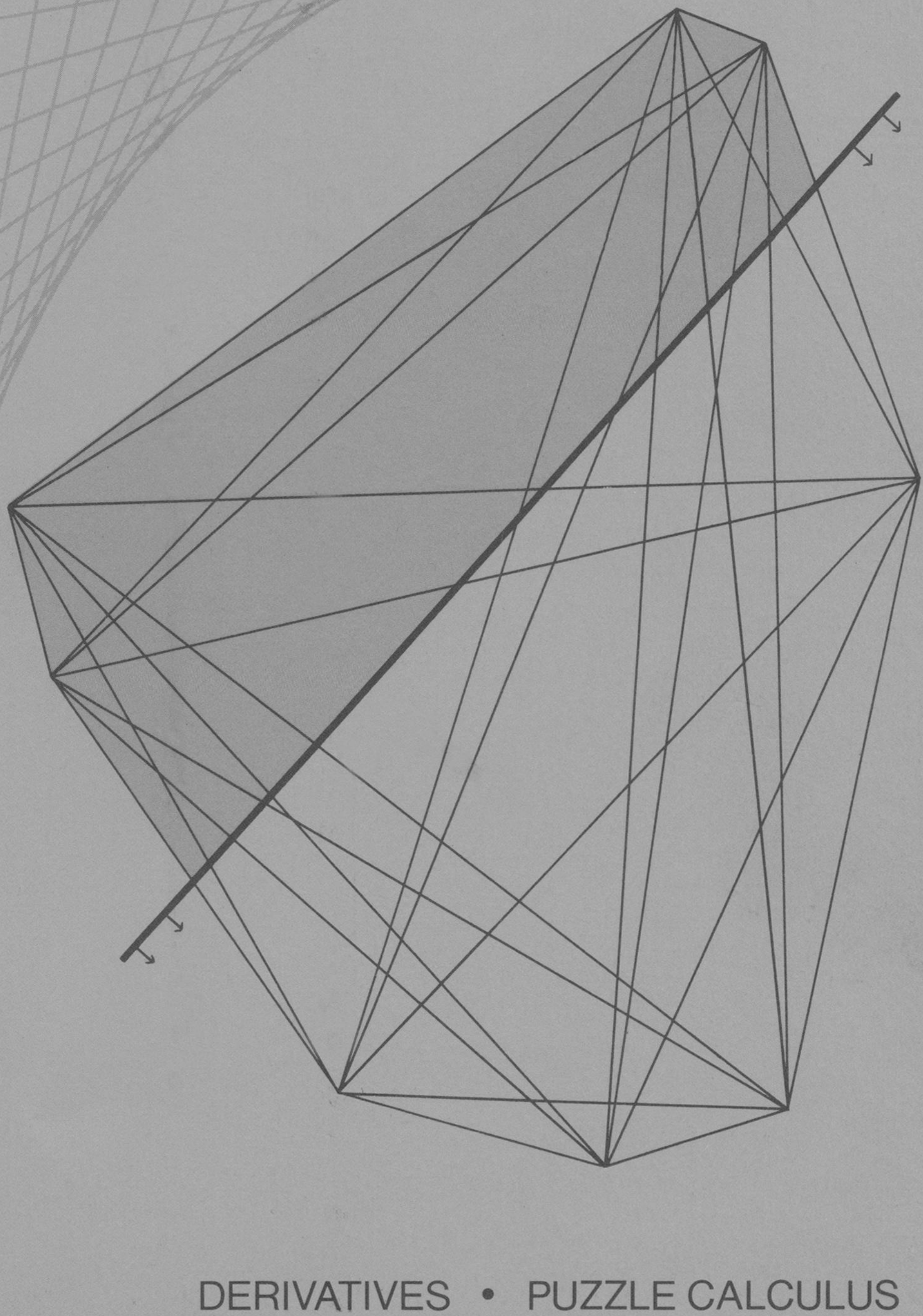

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COVER: The complete polygon on eight vertices being traversed by a counting line. By counting the regions contained in this figure as the line first touches them we can find all 91 regions contained in this figure (see page 23).

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ABOUT OUR AUTHORS

Andrew Bruckner ("Derivatives: Why They Elude Classification") earned both his BA and PhD (1959) at UCLA and since has been on the faculty of the University of California at Santa Barbara. He is the author of over fifty articles on real functions and related topics, including the encyclopedic article "Derivatives" (Amer. Math. Monthly 73 No 4 Part II, 1966), to which his present article is a nice introduction.

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Derivatives:

Why They Elude Classification

The search for functions that can be derivatives leads to several interesting classes of functions and detours around some pathological examples.

ANDREW M. BRUCKNER
University of California, Santa Barbara

1. Introduction

The purpose of this article is to examine, in the specific case of the class of derivatives, a problem that often arises when one studies a class of functions. The problem is that of characterizing the functions in the class. The process usually works like this: one defines a class $\mathscr{C}$ of functions and then proceeds to find out what one can say about the behavior of functions in the class. In the process, one seeks *characterizations* of the class; that is, theorems of the type "$f \in \mathscr{C}$ if and only if f has such-and-such a property". Such characterizations can take a number of different forms. For example, after stating what one means by a continuous function, one might prove the theorem that a function f is continuous if and only if every set of the form $\{x: f(x) < \alpha\}$ or $\{x: f(x) > \alpha\}$ (where α is an arbitrary real number), is an open set. Or, at some point in the development of continuous functions, one might prove Weierstrass's theorem which states that a function defined on a closed interval $[a, b]$ is continuous if and only if it is the uniform limit of a sequence of polynomials defined on $[a, b]$. The first of these theorems gives a characterization in terms of the so-called **associated sets** $\{x: f(x) < \alpha\}$ and $\{x: f(x) > \alpha\}$. The second is an example of a characterization in terms of approximation by simpler functions. Theorems of these types are plentiful in the study of function classes. For example, the class $\mathscr{B}_1$ of functions of **Baire type 1** (which we shall encounter frequently in what follows) is defined as the class of functions which are (pointwise) limits of continuous functions. In this case the very definition involves approximations by a "simpler class" of functions. A characterization of $\mathscr{B}_1$ in terms of associated sets is not difficult to find [**6**, **8**]. A function f is in $\mathscr{B}_1$ if and only if every associated set S is of type F_σ — that is, $S = \bigcup_{k=1}^{\infty} S_k$, with each set S_k being closed. The class $\mathscr{B}_1$ can also be characterized in terms of a concept involving continuity: The function f is in $\mathscr{B}_1$ if and only if its restriction to each perfect subset P of its domain has a point of continuity [**8**] (relative to P).

We mention in passing — without going into detail — two additional classes of functions for which several characterizations are known. The definition of a measurable function involves the associated sets. Lusin's Theorem [**6**, **8**] gives a characterization in terms of simpler (continuous) functions. Another less-known theorem of Lusin [**11**] gives a characterization in terms of derivatives. The (complex) analytic functions are usually defined in terms of differentiability. Standard characterizations involve the Cauchy-Riemann equations, or conformal maps, or representation in power series, or integrals (Cauchy's Theorem).

There have been several attempts at characterizing the class of derivatives. Some of these attempts have not been fully successful, but have succeeded in shedding considerable light on the behavior of derivatives. In the sequel, we shall discuss some of these attempts, point out some of the difficulties, and, in the process, observe some of the strange behavior possible of a derivative.

2. The class of derivatives

Throughout the remainder of this article, we shall be concerned with real functions of a real variable. Such a function f is called a **derivative** provided there exists a function F such that $F'(x)=f(x)$ for *all* x. In the present section, we shall take a look at the behavior of derivatives. We shall see that derivatives can be much more badly behaved than the derivatives one generally meets in courses of elementary (or even advanced) calculus. On the other hand, we shall also see that a derivative must possess some sort of structure which prevents it from being "too pathological". Then, in Section 3 below, we will see how one might try to exploit this "good" behavior of derivatives in order to obtain a characterization of the class of derivatives.

We begin by noting that the derivatives one encounters in elementary calculus are generally very well behaved. In fact, such a derivative is usually continuous except for an occasional point where it is not defined. Then, in advanced calculus, one finds that it is possible for a derivative to be defined at a point without being continuous there. For example, the function $F(x)=x^2\sin(1/x)$ $(F(0)=0)$ is defined for all real x. If $x\neq 0$, the derivative is obtained by standard methods of elementary calculus. For $x=0$, one finds $F'(0)=0$, either directly from the definition of derivative as $F'(x)=\lim_{h\to 0}(F(x+h)-F(x))/h$, or by observing that the graph of F is trapped between the parabolas $y=x^2$ and $y=-x^2$, each of which is tangent to the x-axis at the origin. But one also can readily verify that there are points arbitrarily close to 0 where F' is near 1, while $F'(0)=0$. It follows that F' cannot be continuous at $x=0$. On the other hand, F' is continuous at every point $x\neq 0$, so even this example is not all that pathological.

Just how badly discontinuous can a derivative be? To fully answer this question is beyond the scope of this paper (see **[3]** for a complete answer). But we can provide an example which shows that a derivative can be very discontinuous indeed. Consider for a moment the function $F(x)=(x-r)^{1/3}$, r a real number. This function has a finite derivative for all $x\neq r$, and $F'(r)=\infty$. We shall spread this behavior at $x=r$ over a set containing all the rationals. Towards this end, let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers in [0, 1] and define a function F by $F(x)=\sum_{k=1}^{\infty}(x-r_k)^{1/3}/10^k$. The series converges uniformly to F on [0, 1] (Weierstrass M-test) from which it follows that F is continuous. Furthermore, each term of the series is strictly increasing, so the same is true of F. One can verify (although the computations are tedious **[10]**) that F' is finite and positive at each point for which the sum of the term-by-term differentiated series converges (a dense set of points) and infinite at all other points. In particular $F'(r_k)=\infty$ for all $k=1,2,\ldots$ so this latter set is also dense. Now let $G(x)=F^{-1}(x)$. Then $G'(x)$ is a non-negative number for each x, and is 0 at all points of the form $F(r_k)$, $k=1,2,\ldots$. Since F is continuous, it maps the interval [0, 1] into some interval $[a,b]$, and it maps the dense set of rationals into a dense subset of $[a,b]$. Thus G is a strictly increasing differentiable function whose derivative is zero on a dense set of points, and positive on a different dense set of points. But at any point x for which $G'(x)>0$, G' must be discontinuous because arbitrarily close to x there are points where G' is zero. Thus, G' must be discontinuous on the dense set $\{F(x)\colon F'(x)>0\}$.

Actually, much worse behavior is possible for a derivative, but we shall not go into that now. Let us instead turn to the good behavior that every derivative enjoys. First we note that if f is the derivative of F, then the definition of the derivative gurarantees that for each x, $f(x)=\lim_{h\to 0}[F(x+h)-F(x)]/h$. In particular, by letting h take on the specific values $1/n$, $n=1,2,\ldots$ we see that $f(x)=\lim_{n\to\infty}n^{-1}[F(x+1/n)-F(x)]$. If we define functions f_n by $f_n(x)=$

$n^{-1}[F(x+1/n)-F(x)]$ we see $f(x)=\lim_{n\to\infty} f_n(x)$. Now each of the functions f_n is continuous, so f is the limit of a sequence of continuous functions. This puts f in the class $\mathcal{B}_1$ defined in the Introduction. The second characterization for $\mathcal{B}_1$ given there guarantees that f has a point of continuity in every closed interval. Thus the set of points of continuity of a derivative must at the very least be a dense set. (In the previous example, this set is contained in the set $\{x: G'(x)=0\}$, a dense set.)

Another property shared by all derivatives is the so-called intermediate value property, also called the **Darboux property** in honor of G. Darboux who studied it about one hundred years ago **[5]**. This property can be stated as follows: if $f(x_1)=y_1$, $f(x_2)=y_2$, and y is between y_1 and y_2, then there exists an x between x_1 and x_2 such that $f(x)=y$. In other words, on each interval, f takes on all values between those it takes at the extremities of that interval. One can say all this more easily by saying f maps connected sets onto connected sets.

Now the Darboux property does not, by itself, guarantee good behavior. For example, consider the function defined on $[0, 1]$ as follows. For each $a \in [0, 1]$ write a in its binary form: $a = .a_1a_2a_3\ldots$ where $a_i=0$ or $a_i=1$ for each i. To be specific, when there are two representations for a, choose that one which terminates in a sequence of ones. Now let $f(a)=\limsup(a_1+a_2+\cdots+a_i)/i$. It is not difficult to verify that f takes on *every* value between 0 and 1 on every subinterval of $[0, 1]$. Thus it must have the Darboux property. Of course, the function f is everywhere discontinuous so it cannot be a derivative. In fact, it cannot be of Baire type 1 (although one can show it is of Baire type 2; that is, it is a limit of a sequence of functions of Baire type 1).

This example shows that Darboux functions can be very pathological. But it turns out that if a function has the Darboux property and is also of Baire type 1, it must possess a reasonable amount of good behavior. And every derivative is, as we have seen, such a function. We shall consider the question of comparing this class with the class of derivatives in the next section.

3. Characterization of derivatives

Derivatives, as we have seen, can be much more badly behaved, in general, than those one encounters in calculus courses. Yet derivatives always are of Baire type 1 and always possess the Darboux property. In the sequel we shall consider several attempts at characterizing derivatives, each of which, in some sense, takes this observation as its starting point. Because of this fact, we begin with a brief comparison of the class Δ of derivatives and the class $\mathcal{DB}_1$, of Darboux functions of Baire type 1.

We first observe that the two classes under consideration have, as classes of functions, very different algebraic structures. For example, the sum of two derivatives is again a derivative. (If $f=F'$ and $g=G'$ then $f+g=(F+G)'$.) But the sum of two functions in $\mathcal{DB}_1$, need not be in that class. Consider for example, the functions f and g defined on $[0, 1]$ by $f(x)=\sin(1/x)$ ($f(0)=1$) and $g(x)=-\sin(1/x)$ ($g(0)=0$). Each of these functions is continuous for all x except $x=0$, so each is of Baire type 1 (apply the second characterization of that class in the Introduction). Each of these functions clearly has the Darboux property. But the sum $f+g$ vanishes except at $x=0$, and $(f+g)(0)=1$. Thus $f+g$ does not have the Darboux property and is therefore not in $\mathcal{DB}_1$. (It also follows that at least one of the functions f and g must fail to be a derivative, because the sum of two derivatives is again a derivative and therefore has the Darboux property.)

On the other hand, the class $\mathcal{DB}_1$ is closed under outside compositions with continuous functions. That is, if f is in $\mathcal{DB}_1$ and h is continuous, then $h\circ f$ is in $\mathcal{DB}_1$. This is true because such a composition preserves the Darboux property (easy to verify) as well as the Baire 1 property. (To check the latter assertion we can use either the first or the second characterization we gave for Baire 1 functions in the Introduction.) Yet, the class Δ is not closed under such a composition. In fact, if h is continuous, then $h\circ f$ is a derivative for every derivative f if and only if h is linear, i.e., $h(x)=ax+b$ for some real a, b. This result is due to Choquet **[4]** but its proof is beyond the scope of this article. As an illustration, we can verify for the particular function $h(x)=x^2$ that $h\circ f$ is not a derivative for every dervative f. That is, the square of a derivative need not be a derivative. In fact, suppose both f and f^2 are derivatives on

[0, 1]; then f^2, being non-negative, is the derivative of an increasing function F. But that makes f^2 integrable in the Lebesgue sense, [**6**, **8**] and that guarantees that f is also Lebesgue integrable. Thus, we need only take a derivative which is not Lebesgue integrable to furnish an example of a derivative whose square is not a derivative. The derivative of the function F defined on [0, 1] by $F(x) = x^2 \cos(1/x^2)$ $(F(0) = 0)$ is such a derivative [**8**].

Other differences between the two classes can be found in [**3**]. And a few further differences will develop in our discussion below. But there is one result, due to Maximoff [**7**], which shows that there is a close tie between the two classes. This theorem states that *every* function in $\mathscr{DB}_1$ can be transformed into a derivative by a suitable change of variables. More precisely, if f is in $\mathscr{DB}_1$ on an interval $[a, b]$, there exists a homeomorphism of $[a, b]$ into itself such that the function $f \circ h$ is a derivative. Thus, while the two classes are in some ways quite different, they are in the sense above, topologically equivalent.

With this background we turn now to the question of characterizing the class of derivatives Δ. Let us see what happens if we try to obtain such a characterization in terms of associated sets. First we should say precisely what we mean by such a characterization. We seek a family of sets $\mathscr{S}$ with the property that a function is a derivative if and only if all of the associated sets (i.e., the sets $\{x: f(x) < \alpha\}$ or $\{x: f(x) > \alpha\}$) are in $\mathscr{S}$. (Recall that for the class of continuous functions $\mathscr{S}$ consists of the open sets, while for the class $\mathscr{B}_1$, $\mathscr{S}$ consists of the sets of type F_σ). Where do we start on our attempt to find such a family? A clue can be found in two valid inclusions: firstly, $\Delta \subset \mathscr{DB}_1$ as we observed earlier; secondly, if $\mathscr{C}$ denotes the class of continuous functions, then $\mathscr{C} \subset \Delta$. To see this we need only recall that part of the Fundamental Theorem of Calculus which asserts that a continuous function is the derivative of its integral. Thus we have the inclusions $\mathscr{C} \subset \Delta \subset \mathscr{DB}_1$. Now each of the classes $\mathscr{C}$ and $\mathscr{DB}_1$ can be characterized in terms of associated sets: a function is in $\mathscr{C}$ if and only if each associated set is open, while a function is in $\mathscr{DB}_1$ if and only if each associated set E is of type F_σ and has the property that each of its points is a bilateral point of accumulation of E (i.e., if $x \in E$ then there are sequences $\{x_n\}$ and $\{y_n\}$ of points in E such that $x_n \uparrow x$ and $y_n \downarrow x$) [**12**].

Very loosely stated, the distinction between the two types of associated sets is this: near each point x of an associated set E of a function in $\mathscr{C}$ is an entire open interval contained in E and containing x, whereas near each point x of an associated set of a function in $\mathscr{DB}_1$ there are infinitely many points of E on each side of x. Thus, membership in $\mathscr{C}$ requires considerably more "fatness" of each associated set E near each member of E than does membership in $\mathscr{DB}_1$. We might ask whether we can find an "intermediate notion of fatness" that would correspond to membership in the intermediate class Δ. This approach was used by Zahorski, who in a very deep and penetrating article [**12**] defined several classes of functions $M_0 \supset M_1 \supset \cdots \supset M_5$. Each of these classes was defined in terms of associated sets. As the classes became smaller, the associated sets became "fatter" near each of their members. For M_0, "fatness" was defined exactly as above for the class $\mathscr{DB}_1$. For M_1, the term "bilateral point of accumulation" was replaced with the term "bilateral point of condensation" (i.e., replace "infinite" by "of cardinality of the continuum" in the definition of point of accumulation). This class also turns out to be $\mathscr{DB}_1$. For M_2 the operating term was "positive measure". The conditions M_3 and M_4 were rather complicated and the operating term for M_5 was "point of density". A description of his many results would take us too far afield at this point. Suffice it to say that he showed if one restricted one's attention to bounded functions, the class of derivatives Δ was trapped between the classes M_4 and M_5: $M_5 \subset \Delta \subset M_4$ for bounded functions. Thus he sought a notion of "fatness" intermediate to the corresponding notions for M_4 and M_5. He asked for a class $M_{4.5}$, defined in terms of associated sets which would exactly give the class of bounded derivatives.

He was unsuccessful! In fact, he showed, by a clever argument, that *no such class could exist.* That is, the class of bounded derivatives could not possibly be characterized in terms of associated sets. But he did not settle the case of all derivatives (bounded or not), which he did show was contained in the class M_3. Perhaps it is possible for this class, Δ, to admit of such a characterization. Let us see.

Let us try to attack this problem by looking at it in a somewhat broader context. Suppose some family $\mathscr{F}$ of functions can be characterized in terms of associated sets. That means that there exists a family $\mathscr{S}$ of sets such that $f \in \mathscr{F}$ if and only if each associated set of f is in $\mathscr{S}$. Now let α be any real number, let $f \in \mathscr{F}$ and let h be any continuous strictly increasing function mapping the real line onto itself. Then the set $S \equiv \{x: (h \circ f)(x) < \alpha\} = \{x: f(x) < h^{-1}(\alpha)\}$. Thus $S \in \mathscr{S}$ since $f \in \mathscr{F}$. The same is true for any set of the form $\{x: (h \circ f)(x) > \alpha\}$. Therefore the function $h \circ f$ must also be in $\mathscr{F}$. In other words, the class must be closed under composition on the outside by continuous strictly increasing functions if it has any chance at all of being characterizable in terms of associated sets. But we saw at the beginning of this section that this was not the case with the class Δ. Thus Δ cannot be characterized in this way. The same is true of many other classes of functions **[2]**.

The fact that this attempt at characterizing derivatives failed does not in any way suggest that the attempt was not worthwhile. Zahorski was able to obtain many deep and far reaching results. Even without making a careful count, one can safely assert that literally hundreds of research investigations by a number of different authors have been directly or indirectly influenced by the results Zahorski did obtain and the questions he raised in his article **[12]**.

Let us turn to a different approach towards a characterization of derivatives. This approach takes its cue from the analogous problem of characterizing (Lebesgue) integrals. A student enrolled in a course which includes an introduction to the Lebesgue integral soon learns that a function F is an integral if and only if it is absolutely continuous. The definition of absolute continuity is generally given in terms of δ's and ε's: the function F is **absolutely continuous** on $[a, b]$ provided for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\Sigma_{k=1}^{n} |F(b_k) - F(a_k)| < \varepsilon$ whenever $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$ is a finite sequence of non-overlapping intervals contained in $[a, b]$ for which $\Sigma_{k=1}^{n} |b_k - a_k| < \delta$. Thus, the definition of absolute continuity differs from the definition of continuity (or uniform continuity) in only one way, a very important way, however. Instead of requiring only that F "grow" very little on small intervals, it requires also that F grow very little on small sets made up of a finite number of non-overlapping intervals whose union is small. These intervals could, for example, be spread out over the interval $[a, b]$ in any manner one wishes, as long as they do not overlap.

One then proves the characterization of integrals that we mentioned: a function F is an integral, that is, there exists a function f defined on $[a, b]$ such that $F(x) = \int_a^x f(t)dt$ for all x in $[a, b]$, if and only if F is absolutely continuous. Thus, we have a characterization of the class of integrals in terms of a concept involving δ's and ε's.

But the absolutely continuous functions, and therefore the integrals, can be characterized in another way that suggests an approach to characterizing derivatives. Two properties of absolutely continuous functions are readily obtained: each absolutely continuous function is both continuous and of bounded variation. But there are continuous functions of bounded variation which are not absolutely continuous. The Cantor function **[6, 8]** is such a function. This function is continuous and monotonically increasing on [0, 1]. It is constant on each interval contiguous to the Cantor set. Thus it maps all points of [0, 1] which are not in the Cantor set onto a countable set. Since it maps [0, 1] onto [0, 1], it maps the zero measure Cantor set onto a set of measure one (actually onto all of [0, 1]). It is easy to verify from the definition of absolute continuity, that this cannot happen for an absolutely continuous function. Such a function must map zero measure sets onto zero measure sets. And it turns out that it is exactly this property that picks out the absolutely continuous functions from among the larger class of continuous functions of bounded variation **[8]**.

We thus have this situation. There is the class $\mathscr{I}$ of integrals (or equivalently, the class of absolutely continuous functions), and also the class $\mathscr{CV}$ of functions which are continuous and of bounded variation. The inclusion $\mathscr{I} \subset \mathscr{CV}$ is proper. If we ask for another condition or restriction on the class $\mathscr{CV}$ which will give us *exactly* the class $\mathscr{I}$ we arrive at the class $\mathscr{N}$ of functions which map all zero measure sets onto zero measure sets. Thus $\mathscr{I} = \mathscr{CVN}$. We have obtained an additional condition, which in the presence of continuity and bounded variation, gives us absolute continuity.

An analogous situation arises when we consider the class Δ of derivatives. As we saw before, each derivative has the Darboux property and is of Baire type 1: $\Delta \subset \mathscr{DB}_1$. The inclusion is proper. We seek another restriction on the class $\mathcal{DB}_1$ which will give us *exactly* the class Δ. Neugebauer **[9]** used the following approach. He first noted that the class $\mathcal{DB}_1$ admitted a characterization which is somewhat different in nature from the kinds of characterizations we have already seen. Let $\{I\}$ denote the class of all closed intervals contained in [0, 1]. We use the notation $I \to x$ to mean, roughly, that I is an interval contracting to x. More precisely, the notation means that each interval of a sequence (or net) of intervals contains x and the lengths of the intervals approach 0. We denote the length of an interval I by $|I|$. Neugebauer proved that a function f is in $\mathcal{DB}_1$ if and only if to each closed interval I there corresponds a point x_I in the interior of I such that $f(x_I)$ converges to $f(x)$ whenever $I \to x$: In symbols

$$f(x_I) \to f(x) \text{ whenever } I \to x. \tag{1}$$

We seek an additional restriction which together with (1) will give us exactly the class Δ.

Suppose now that F is differentiable on [0, 1] and let $f = F'$. The Mean Value Theorem guarantees that to each interval $[a, b]$ there corresponds a point $x_0 \in (a, b)$ such that $(F(b) - F(a))/(b-a) = F'(x_0) = f(x_0)$. Thus, with each interval $I = [a, b]$ we can associate a point, which we shall now denote by x_I, such that $F(b) - F(a) = f(x_I)|I|$. It is easy to verify that if I and J are two abutting intervals (say $I = [a, b]$ and $J = [b, c]$), then

$$f(x_{I \cup J}) = \frac{f(x_I)|I| + f(x_J)|J|}{|I \cup J|}. \tag{2}$$

(This somewhat strange looking equality simply states that the interval function $f(x_I)|I|$ is additive.) We can also check readily that f also meets condition (1) above. Conversely, any function meeting the conditions (1) and (2) must be a derivative. Stated more precisely, this means that if we can associate with each interval I a point x_I in the interior of I such that conditions (1) and (2) are met, then there is a differentiable function F such that $F' = f$. To see this, we define F as follows: we let $F(0) = 0$, and for $0 < x \leqq 1$ we let $F(x) = f(x_I)|I|$ where $I = [0, x]$. It follows from (2) that for $I = [a, b]$, $F(b) - F(a) = f(x_I)(b-a)$. And it follows from (1) that if $I \to x$, $(F(b) - F(a))/(b-a) \to f(x)$. Thus $F'(x) = f(x)$, and f is in Δ.

What this gives us is a characterization of derivatives in terms of a condition (condition (2)) which picks out the functions in Δ from among the functions in $\mathcal{DB}_1$ in much the same manner as the condition $\mathcal{N}$ picked out the integrals from among the continuous functions of bounded variation.

As we stated earlier, this characterization is somewhat different in nature from the other characterizations we encountered. And it does not seem to supply a very practical test (but that often happens). To apply the test to a specific function we would have to determine whether one can pick one point x_I out of each interval I, such that conditions (1) and (2) are met, and the proof does not suggest any way of doing this. Nonetheless, it gives us the kind of characterization we sought. And we still are free to look for other such characterizations.

Another approach to the problem of characterizing derivatives was used by I. Maximoff in a series of articles. (A unification of his work, along with further results can be found in Agronsky **[1]**.) Unfortunately, this approach is extremely complicated, so we shall restrict ourselves to a very brief outline.

Roughly speaking, here is the approach. Suppose we say the set A is **strongly contained** in the set B provided every point of A is a bilateral condensation point of B. We can then write each associated set of a function f in $\mathcal{DB}_1$ as a union of an expanding sequence of perfect sets, each strongly contained in its successor. This can be done in such a way as to guarantee also that if $\beta > \alpha$ then each perfect set associated with the set $\{x: f(x) < \alpha\}$ is also strongly contained in the corresponding perfect set associated with $\{x: f(x) < \beta\}$. (The same is true for the sets where $f > \alpha$ and $f > \beta$.) It turns out that the converse is true: if there exists such a family of perfect sets, f must be in $\mathcal{DB}_1$. Now, we can vary our definition of strong containment. The one we just gave corresponds to functions in $\mathcal{DB}_1$ (equivalently Zahorski's class M_1). For example, if we require more of our notion of strong containment, namely that each point of A be a point of density of B, we arrive at Zahorski's class M_5 (the so-called approximately continuous functions). Thus we can ask if there exists a suitable notion of strong containment which gives rise exactly to the class of derivatives. Agronsky **[1]** was able to obtain strong containment properties which corresponded to each of the Zahorski's classes M_1, M_2, M_3, M_4 and M_5. But none for the class of derivatives. It is not clear whether such a characterization is possible for the class Δ. It might be impossible, as it was impossible to characterize the class Δ in terms of associated sets. The question is one worth pursuing.

4. Conclusion

In the preceding sections, we have tried to look at some of the approaches used in trying to find characterizations of the class of derivatives. In the process, we have tried to give indications of some of the odd behavior possible of derivatives: badly discontinuous derivatives, derivatives whose squares are not derivatives and the like. But we also saw that all derivatives shared certain regularizing properties: each derivative is of Baire type 1 and has the Darboux property. There is much more known about derivatives, some good and some bad. (A fuller study of the behavior of derivatives and related classes of functions can be found in **[3]**.) Many new results giving further information appear in the mathematical journals each year. The problem of characterizing certain types of generalized derivatives also appears to be unsolved (and difficult!). For example, we do not know of any characterizations for any of the following types of functions: Dini derivatives, approximate derivatives, Peano derivatives and symmetric derivatives. Each of these classes is important in certain parts of mathematics, and none has enjoyed a characterization. And one can ask the same questions about various types of derivatives of functions of several variables. These problems all appear to be difficult and may escape solution for a long time.

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I never could understand *Principia*

Symbolism is useful because it makes things difficult. Now in the beginning everything is self-evident, and it is hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy to correctness. Hence we must invent a new and difficult symbolism in which nothing is obvious.

— BERTRAND RUSSELL

A Calculus for Know/Don't Know Problems

A symbolic model for a class of popular puzzles leads to a solution algorithm.

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1. Introduction

Problems of the know/don't know type usually involve people wearing hats, having stamps stuck on their foreheads or numbers pinned on their backs. They are asked in turn if they know what their own object, which they cannot see, is. A popular version (see, for example, [1]) proceeds as follows. After a number of "don't know" answers, one person is able to deduce from these answers and the fact that everyone can see every other person's object, what his own object is. Here is a typical example.

Two players P_1 and P_2 know that the questionmaster X has four red stamps and three blue stamps. The questionmaster sticks three stamps on each player and tells them that each has at least one red stuck on him. P_1 says he doesn't know his stamps and then P_2 says he doesn't know his. Finally P_1 says he does know his. What stamps were stuck on P_1?

P_1 discovered his stamps by the following deduction: "I have *RRR* or *RRB* or *RBB*. If I had *RRR* then P_2 would have known that he had *RBB* and he would have said so. As he did not, it follows that I do not have *RRR*. If I had *RBB* then P_2 would say to himself, after my first answer. 'As P_1 has *RBB* I have *RRR* or *RRB* but if I had *RRR* then P_1's first answer would be that he knew his stamps, which is not the case. So I have *RRB*.' As P_2 said that he did not know his stamps it follows that I do not have *RBB*. Thus I must have *RRB*."

It is clear that the arguments involved become very complicated if any larger number of replies are involved and so the aim of this paper is to give a calculus which allows such situations to be analyzed without a large number of "if he knew that he knew that he knew ..." 's.

2. A General Setting

We begin by giving a general method of expressing these problems in terms of sets. Let X be the questionmaster and let $P_1, \ldots, P_m$ be the players. We first form the finite set U of all the possible situations which are consistent with the information in the problem. In our example $U = \{P_1 = RRR \,\&\, P_2 = RBB, P_1 = RRB \,\&\, P_2 = RRB, P_1 = RRB \,\&\, P_2 = RBB, P_1 = RBB \,\&\, P_2 = RRR, P_1 = RBB \,\&\, P_2 = RRB\}$ which we will denote by $\{a_1, a_2, a_3, a_4, a_5\}$ respectively. From $U = \{a_1, a_2, \ldots, a_n\}$, X secretly selects an element a. In our example a is the situation which X has produced by sticking on the stamps.

Since P_1 can see the stamps on P_2, he knows, grouping the situations according to the states of P_2 which he might possibly observe, which of the following alternatives is the case: $a \in \{a_1, a_3\}$ or $a \in \{a_2, a_5\}$ or $a \in \{a_4\}$. A similar remark applies to P_2: he can partition U according to the possible states of P_1 and determine, based on observation of P_1, which element of the partition a is in. In general, each P_i may partition U into a family $I_i = \{I_{i1}, \dots, I_{im_i}\}$ of subsets of U grouped according to the possible states of the other players. (In our example $I_1 = \{\{a_1, a_3\}, \{a_2, a_5\}, \{a_4\}\}$, $I_2 = \{\{a_1\}, \{a_2, a_3\}, \{a_4, a_5\}\}$.) P_i's observation of the other players enables him to determine which I_{ij} the chosen situation a is actually in.

Now in our example P_1 is asked if he knows what his own stamps are, or, in other words, if he knows which of the following alternatives is the case: $a \in \{a_1\}$ or $a \in \{a_2, a_3\}$ or $a \in \{a_4, a_5\}$. This question leads each P_i, in general, to a different partition of U, $Q_i = \{Q_{i1}, \dots, Q_{in_i}\}$, where P_i is asked if he knows which Q_{ij} a is in. In our example $Q_1 = \{\{a_1\}, \{a_2, a_3\}, \{a_4, a_5\}\}$, $Q_2 = \{\{a_1, a_3\}, \{a_2, a_5\}, \{a_4\}\}$.

This first part of the method, viz., listing U and the I_i and the Q_i, amounts to translating the verbal description of the problem into an explicit working form. At this point of the analysis each of the P_i's knows all the I_j's and Q_j's. Now let $S_1, S_2, S_3, \dots$ be a sequence such that each S_i is one of the P_j's, possibly with repetitions. In turn X asks each of $S_1, S_2, S_3, \dots$ his question and receives an answer which all of the P_j's hear. In our example the S-sequence is P_1, P_2, P_1.

There are two possible problems. In one we are given the sequence $A_1, A_2, A_3, \dots$ of yes/no answers of $S_1, S_2, S_3, \dots$ and we have to find a or some information about a. In the other we are given a and have to find the A-sequence $A_1, A_2, A_3, \dots$. In the above example the A-sequence is No, No, Yes and we had to deduce which Q_{ij} contains a.

3. The Calculus

Suppose first that we are given the A-sequence. For $k = 0, 1, 2, \dots$ define the subset V_k of U to consist of those elements of U which if they were each a would produce the answers $A_1, A_2, \dots, A_k$. The V_k are calculated inductively in the following way. (We assume that the P_j's can carry out the following calculations.)

Start with $V_0 = U$. We construct V_k from V_{k-1} and the answer A_k given by S_k. Take each a_j in V_{k-1} in turn: Suppose $a = a_j$. Now if $S_k = P_i$ then P_i will know a is in I_{ih}, say. In fact he will know a is in $I_{ih} \cap V_{k-1}$. If $I_{ih} \cap V_{k-1}$ is a subset of some Q_{ip} then A_k will be "yes", otherwise it will be "no". If this agrees with the actual A_k given then a_j is a possibility for a and so a_j goes into V_k, otherwise a_j does not go into V_k. The set V_k consists of all the a_j obtained in this way.

The V-sequence so obtained is used to answer the question about a. Later examples give information about the "convergence" of the V-sequence.

Suppose now that we are given a. To find the A-sequence from the V-sequence we construct the following in order: $V_0, A_1, V_1, A_2, V_2, \dots$. The above procedure shows how to construct V_k from V_{k-1} and A_k. To construct A_k from V_{k-1} and a we work as follows: Suppose $S_k = P_i$ and $a \in I_{ih}$, then if $I_{ih} \cap V_{k-1}$ is a subset of some Q_{ip} then A_k is "yes", otherwise it is "no".

Constructing the V-sequence for our paradigm we obtain: $V_0 = U$; $A_1 = \text{no}$; $V_1 = \{a_1, a_2, a_3, a_5\}$; $A_2 = \text{no}$; $V_2 = \{a_2, a_3\}$; $A_3 = \text{yes}$; $V_3 = \{a_2, a_3\}$. Thus the possible situations which give rise to the specified answers are $a_2 = (P_1 = RRB \;\&\; P_2 = RRB)$ and $a_3 = (P_1 = RRB \;\&\; P_2 = RBB)$, so P_1 does indeed know that $P_1 = RRB$.

4. Applications

Our first application is based on a problem occurring in a school textbook **[2]**. Four boys are standing in a row so that each boy can see all the boys in front of him. Thus P_1 can see P_2, P_3 and P_4 whereas P_4 can see none of the others. There are four red hats and one blue hat. A hat is placed on each boy's head. They are asked in turn if they know the color of their own hat. First P_1, then P_2 and then P_3 all answer yes. What is P_4's answer? Here we have:

$U=\{a_1=P_1$ has the blue hat, $\quad I_1=\{\{a_1,a_5\},\{a_2\},\{a_3\},\{a_4\}\}, \quad Q_1=\{\{a_1\},\{a_2,a_3,a_4,a_5\}\},$

$a_2=P_2$ has the blue hat, $\quad I_2=\{\{a_1,a_2,a_5\},\{a_3\},\{a_4\}\}, \quad Q_2=\{\{a_1,a_3,a_4,a_5\},\{a_2\}\},$

$a_3=P_3$ has the blue hat, $\quad I_3=\{\{a_1,a_2,a_3,a_5\},\{a_4\}\}, \quad Q_3=\{\{a_1,a_2,a_4,a_5\},\{a_3\}\},$

$a_4=P_4$ has the blue hat, $\quad I_4=\{\{a_1,a_2,a_3,a_4,a_5\}\}; \quad Q_4=\{\{a_1,a_2,a_3,a_5\},\{a_4\}\}.$

$a_5=$ everyone has a red hat$\}$;

Applying the calculus we obtain the following V-sequence: $V_0=U$, $A_1=$ yes, $V_1=\{a_2,a_3,a_4\}$, $A_2=$ yes, $V_2=\{a_2,a_3,a_4\}$, $A_3=$ yes, $V_3=\{a_4\}$. Thus there is only one situation possible and so P_4's answer is yes, i.e., the V-sequence "converges".

The second application consists of a problem supposedly considered by Conway and Patterson at a Congress in Moscow, but, as far as I know, without any published results: A card bearing a positive integer is pinned on the back of each of P_1 and P_2 and they are told this and the fact that the sum of the integers is 6 or 7. Each can see the other's integer. They are asked in turn, starting with P_1, if they know what their own integer is. If a three is placed on the back of each, what is the sequence of answers?

Setting up our calculus we have $U=\{15,24,33,42,51,16,25,34,43,52,61\}$, where each two digit number ab means "P_1 has a and P_2 has b." Then

$$I_1=Q_2=\{\{15,25\},\{24,34\},\{33,43\},\{42,52\},\{51,61\},\{16\}\},$$

$$I_2=Q_1=\{\{15,16\},\{24,25\},\{33,34\},\{42,43\},\{51,52\},\{61\}\}.$$

Now $a=33$. The S-sequence is $P_1,P_2,P_1,P_2,\ldots$. Applying the calculus we obtain the following V,A-sequence:

$$V_0=U,\quad S_1=P_1,\quad I_{1h}=\{33,43\}=I_{1h}\cap V_0\not\subseteq Q_{1p} \text{ for all } p,$$

so $A_1=$ no;

$$V_1=\{15,24,33,42,51,25,34,43,52,61\},\quad S_2=P_2,\quad I_{2h}=\{33,34\}=I_{2h}\cap V_1\not\subseteq Q_{2p} \text{ for all } p,$$

so $A_2=$ no;

$$V_2=\{24,33,42,51,25,34,43,52\},\quad S_3=P_1,\quad I_{1h}=\{33,43\}=I_{1h}\cap V_2\not\subseteq Q_{1p} \text{ for all } p$$

so $A_3=$ no;

$$V_3=\{24,33,42,34,43,52\},\quad S_4=P_2,\quad I_{2h}=\{33,34\}=I_{2h}\cap V_3\not\subseteq Q_{2p} \text{ for all } p,$$

so $A_4=$ no;

$$V_4=\{33,42,34,43\},\quad S_5=P_1,\quad I_{1h}=\{33,43\}=I_{1h}\cap V_4\not\subseteq Q_{1p} \text{ for all } p,$$

so $A_5=$ no;

$$V_5=\{33,43\},\quad S_6=P_2,\quad I_{2h}=\{33,34\},\quad I_{2h}\cap V_5=\{33\}\subseteq Q_{2p}=\{33,43\},$$

so $A_6=$ yes;

$$V_6=\{33,43\},\quad S_7=P_1,\quad I_{1h}=\{33,43\}=I_{1h}\cap V_6\not\subseteq Q_{1p} \text{ for all } p,$$

so $A_7=$ no. This pattern will continue with P_2 answering yes and P_1 answering no. So the sequence does not converge and complete knowledge is not possible.

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A Ruin Problem

An interpretation of ruin problems as random walks on polygons with the aim of visiting all the vertices.

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1. Introduction

In the classical ruin problem two players with finite initial capitals play until one of them is ruined. It is often useful in analyzing this problem, for instance in finding the probability that a specific player is ruined and the expected duration of the game, to interpret the game as a random walk on the line with absorbing barriers. We will here treat a ruin problem which can be interpreted as a random walk on a polygon. We start with two simple cases.

2. Two simple cases

A particle performs a random walk between the vertices of a triangle by making successive steps to adjacent vertices. In each step the particle moves counterclockwise with probability p and clockwise with probability q, $p+q=1$. The random walk continues until all three vertices of the triangle have been visited. We are interested in the expected duration of the random walk and the probability that the step, which finishes the walk, is in the counterclockwise direction.

This random walk is equivalent to the following ruin problem. Two players A and B make a sequence of plays. In each play A wins with probability p and B with probability q, $p+q=1$. The loser in a play gives a dollar to the winner. At the start they have one dollar each. Furthermore there is a reserve capital of one dollar. The player who loses the first play is allowed to use this dollar for continued playing until one of the two players is ruined. This dollar, however, is never returned to the reserve.

The random walk and the associated ruin play are described by the graph in FIGURE 1. In the random walk interpretation the state "B is finally ruined" corresponds to the state R meaning that the random walk on the triangle is finished by a step in the counterclockwise direction and the state "A is finally ruined" corresponds to the state L meaning that the walk ends with a step in the clockwise direction.

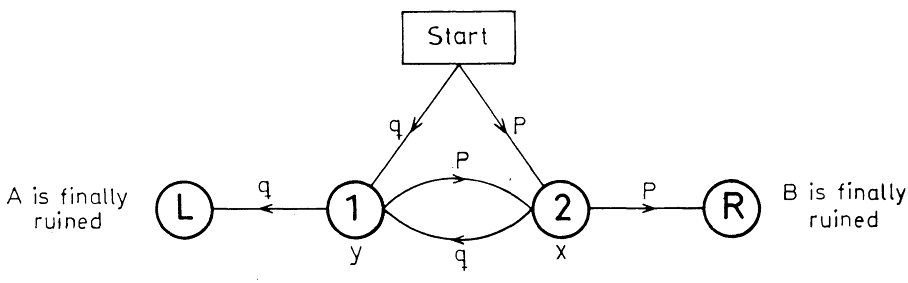

FIGURE 1.

By ξ_3 we denote the duration of the play and by r_3 and s_3 we denote the probabilities that the play ends with ruin of B and A respectively, or r_3 and s_3 are the probabilities for absorption in states R and L respectively. Let y and x be the probabilities for absorption in state R from states 1 and 2 as indicated in FIGURE 1. From the total probability formula we then obtain $r_3 = px + qy$, $x = p + qy$, and $y = px$, which gives $r_3 = (1+q)p^2/(1-pq)$. By symmetry we then obtain $s_3 = (1+p)q^2/(1-pq)$. Note that $r_3 + s_3 = 1$.

If y and x instead are the expected durations from state 1 and 2 we obtain by conditioning $E[\xi_3] = 1 + px + qy$, $x = 1 + qy$, and $y = 1 + px$. This gives $E[\xi_3] = (2 + pq)/(1 - pq)$. Furthermore, let η_3 be the number of times that A is ruined during the game (possible values of η are 0, 1, and 2). It follows that $E[\eta_3] = q + s_3 = q(1+q)/(1-pq)$.

Now consider the corresponding random walk on a square until all vertices have been visited. This random walk is equivalent to a gambling situation as above, but in this case the reserve capital is 2 dollars. So the players have two possibilities to take one dollar, when ruined, from the reserve capital in order to continue the game. This random walk and this game are described by the graph in FIGURE 2. Note that the times when dollars are taken from the reserve corresponds in the random walk interpretation to the times when a vertex of the square is first visited.

Let ξ_4 be the duration of the play and let r_4 and s_4 be the probabilities for ultimate ruin of B and A respectively. With the same method as above we then obtain

$$r_4 = \frac{p^3(1+2q^2)}{(1-pq)(1-2pq)}, \quad s_4 = \frac{q^3(1+2p^2)}{(1-pq)(1-2pq)}, \quad E[\xi_4] = \frac{3-4pq+4p^2q^2}{(1-pq)(1-2pq)}.$$

Let η_4 be the number of times that A is ruined during the game. Then $E[\xi_4] = q + s_3 + s_4$, which gives

$$E[\xi_4] = \frac{q(1+q+q^2-2pq(1+q^2))}{(1-pq)(1-2pq)}.$$

3. The general case

It is easily seen that the gambling described by the graph in FIGURE 2 has the following structure. The players go through a sequence of classical ruin situations starting with 1 and 1, 1 and 2 and 1 and 3 dollars. In the corresponding random walk on a square these situations correspond to the random walks between times when a new vertex is visited. This structure can be used in connection with the general case.

Thus we consider a gambling situation as above, where the players A and B have probabilities p and q of winning in each play. They start with one dollar each and a reserve capital of $n-2$ dollars. As

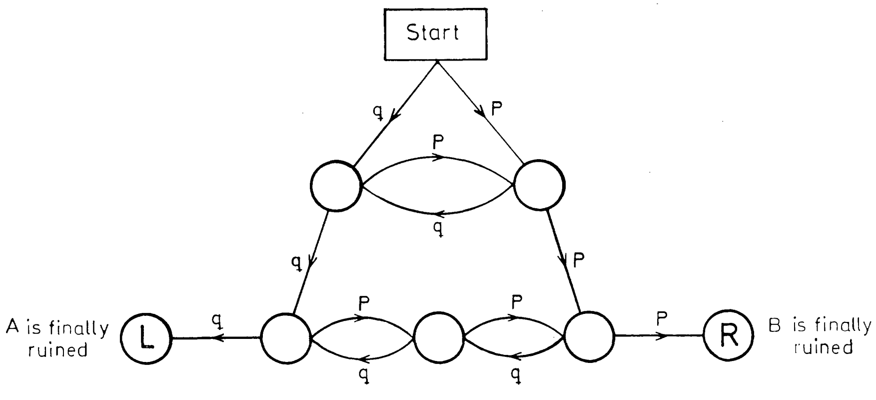

FIGURE 2.

TABLE 1

n	p	r_n	$E[\xi_n]$	$E[\eta_n]$
3	0.10	0.02	2.30	1.88
3	0.45	0.42	2.99	1.13
3	0.60	0.66	2.95	0.74
5	0.10	0.00	4.84	3.88
5	0.55	0.65	9.80	1.61
5	0.80	0.99	6.05	0.33
8	0.20	0.00	11.1	6.67
8	0.45	0.27	26.4	4.51
8	0.90	1.00	8.6	0.13
10	0.10	0.00	11.1	8.88
10	0.55	0.78	41.1	2.95
10	0.80	1.00	14.4	0.33

RUIN PROBLEM WITH RESERVE CAPITAL: Total capital of n dollars is used to establish a game in which two players A and B compete until one or the other is ruined. Each player is given one dollar, the remaining $n-2$ dollars being held in reserve. On each play A wins with probability p, and B wins with probability $1-p=q$; in each case the loser pays the winner one dollar. Whenever either player is temporarily ruined (i.e., reduced to 0 dollars) he can draw one dollar from the reserve in order to continue play. Final ruin occurs when the reserve is gone and one player has no more money. This table illustrates numerical values of the probability r_n of ruin of player B, expected duration $E[\xi_n]$ and expected number of times $E[\eta_n]$ that A is ruined.

long as the reserve capital exists, a ruined player may take one dollar from it and use this dollar for continued gambling. A dollar taken from the reserve is never returned to it. By ξ_n we denote the duration of the game and by r_n and s_n we denote the probabilities that B and A respectively are finally ruined.

This game is equivalent to a random walk on a polygon with n vertices, where steps occur counterclockwise with probability p and clockwise with probability q. The random walk continues until all vertices have been visited.

We will use the following results from the classical ruin problem (Feller [1]), where players A and B start with z and $n-z$ dollars and play until one of the players is ruined. If $p_n(z)$ is the probability that B is ruined and $E_n(z)$ is the expected duration, then

$$p_n(z) = \frac{(q/p)^z - 1}{(q/p)^n - 1}, \qquad p \neq q$$

$$E_n(z) = \frac{z}{q-p} - \frac{n}{q-p} \cdot \frac{(q/p)^z - 1}{(q/p)^n - 1}, \qquad p \neq q \tag{1}$$

$$E_n(z) = z(n-z) \qquad p = q = 1/2.$$

We can then determine r_n from the recursive formulas

$$r_n = r_{n-1}\, p_n(n-1) + (1 - r_{n-1})\, p_n(1), \qquad r_3 = (1+q)p^2/(1-pq).$$

This gives for $p \neq q$

$$r_n = \frac{p^{n-1}[p^n - q^n - nq^{n-1}(p-q)]}{(p^n - q^n)(p^{n-1} - q^{n-1})}, \qquad n = 3, 4, 5, \ldots. \tag{2}$$

An expression for s_n is obtained by switching p and q in (2). Note that $r_n + s_n = 1$. It follows easily from the above that $r_{n+1} > r_n$ for $p > 1/2$. Thus it is the better player who benefits from the reserve. Also note that $p > 1/2$ implies $\lim_{n\to\infty} r_n = 1$.

There seems to be no simple expression for the expected duration $E[\xi_n]$, but $E[\xi_n]$ can be calculated from the recursion

$$E[\xi_n] = E[\xi_{n-1}] + r_{n-1} E_n(n-1) + s_{n-1} E_n(1), \qquad n = 4, 5, 6, \ldots \tag{3}$$
$$E[\xi_3] = (2 + pq)/(1 - pq).$$

Furthermore, let η_n be the number of times that the player A is ruined during the game. Then $E[\eta_n] = q + s_3 + s_4 + \cdots + s_n$, $n = 3, 4, 5, \ldots$. Numerical values of these formulae are provided in TABLE 1 for various values of n and p.

We have assumed in the derivations above that $p \neq q$. For the case $p = q = 1/2$ it follows immediately from symmetry that $r_n = s_n = 1/2$ and from (3) with $E_n(z) = z(n - z)$ we obtain

$$E[\xi_n] = 1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2} = \binom{n}{2}.$$

This formula is given in F. Göbel-A. A. Jagers **[2]** and in Råde **[3]**.

There are many obvious variations of the ruin problem treated above. The players can for instance start with initial capitals a and b and a reserve capital of c dollars. They can also be allowed to take more than one dollar, when ruined, from the reserve capital.

The author would like to thank Nils-Gunnar Pehrsson, for the calculation of TABLE 1, and the referee for constructive and helpful comments.

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[2] F. Göbel and A. A. Jagers, Random walks on graphs, stochastic processes and their applications, 2 (1974) 311–336.

[3] L. Råde, Random walks on a hexagon, Int. J. Math. Educ. Sci. Technol., 6 (1975) 255–263.

The way it used to be

It may be observed of mathematicians that they only meddle with such things as are certain, passing by those that are doubtful and unknown. They profess not to know all things, neither do they affect to speak of all things. What they know to be true, and can make good by invincible arguments, that they publish and insert among their theorems. Of other things they are silent and pass no judgment at all, choosing rather to acknowledge their ignorance, than affirm anything rashly. They affirm nothing among their arguments or assertions which is not most manifestly known and examined with utmost rigour, rejecting all probable conjectures and little witticisms. They submit nothing to authority, indulge no affection, detest subterfuges of words, and declare their sentiments, as in a court of justice, *without passion, without apology*; knowing that their reasons, as Seneca testifies of them, are not brought to *persuade*, but to compel.

— ISAAC BARROW

Factoring Integers Whose Digits Are All Ones

Some factors of a special class of integers which includes the Mersenne primes.

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Let b be a positive integer greater than one. For each positive integer k, define $I_k = b^{k-1} + b^{k-2} + \cdots + b + 1$. Thus I_k is the integer whose base b representation consists of k ones. If $(n, b) = 1$, i.e., if n and b are relatively prime, we define $c(n)$ to be the smallest positive integer k such that $n \mid I_k$ (n divides I_k). In our first proposition, we will show that if $(n, b) = 1$, then $c(n)$ exists and furthermore $c(n) \leqq n$. Clearly if n and b are not relatively prime, then $n \nmid I_k$ for any positive integer k. In this paper, we characterize those n for which $c(n) = n$ and those for which $c(n) = n - 1$. We prove that if $b \not\equiv 3 \pmod 4$, then $c(n) = n$ if and only if each prime factor of n is also a divisor of $b - 1$. In particular, if $b = 10$, then $c(n) = n$ if and only if n is a power of 3 — a fact conjectured by Sobczyk [1]. If $b \equiv 3 \pmod 4$, then the condition $4 \nmid n$ must be added. Finally we prove $c(n) = n - 1$ if and only if n is a prime which does not divide $b - 1$, and b is a primitive root of n.

If n is a prime, it is easy to determine $c(n)$ (Propositions 2 and 6). Hence, we show (Proposition 5) that it suffices to know $c(n)$ whenever n is a power of a prime. Then we prove (Propositions 7 and 9) that $c(p^j)$, $j \geqq 3$, can be determined inductively from the values of $c(p)$ and $c(p^2)$. The value of $c(p^2)$ is given by Proposition 8. Our main results are stated as Propositions 11 and 12. An interesting question which we leave unanswered is, "For which primes p is I_p prime?" That this is indeed a difficult question is clear because when $b = 2$, this is exactly the problem of determining the Mersenne primes.

PROPOSITION 1. *For each positive integer n satisfying $(n, b) = 1$, there exists a positive integer $k \leqq n$ such that $n \mid I_k$.*

Proof. At least two of the $n + 1$ integers in the sequence $I_1, I_2, \ldots, I_n, I_{n+1}$ must belong to the same residue class modulo n, say I_i and I_j where $1 \leqq i < j \leqq n + 1$. Thus

$$b^i I_{j-i} = I_j - I_i \equiv 0 \pmod n.$$

Since $(n, b) = 1$, we have $n \mid I_{j-i}$ and $1 \leqq j - i \leqq n$. The proposition is proved.

PROPOSITION 2. *If $(n, b) = (n, b - 1) = 1$ and $n > 1$, then*
(i) $n \mid I_k$ *if and only if* $b^k \equiv 1 \pmod n$; *and*
(ii) $c(n) < n$.

Proof. Since $(b - 1) I_k = b^k - 1$ and since, by hypothesis, $(n, b - 1) = 1$, the truth of (i) is evident. It remains to show $c(n) < n$. Now $1, b, b^2, \ldots, b^{n-1}$ represent nonzero residues modulo n. Since there are only $n - 1$ distinct nonzero residues modulo n, we must have $b^i \equiv b^j \pmod n$ for some i and j, $0 \leqq i < j \leqq n - 1$. Because $(n, b) = 1$, this is equivalent to $b^{j-i} \equiv 1 \pmod n$. By (i), $n \mid I_{j-i}$ and $1 \leqq j - i \leqq n - 1$. Thus $c(n) \leqq n - 1 < n$.

PROPOSITION 3. *If m and n are positive integers, we have*
(i) $I_{mn} = I_n(b^{(m-1)n} + b^{(m-2)n} + \cdots + b^n + 1)$;
and
(ii) $I_{mn} - m I_n = I_n(b - 1)(I_{(m-1)n} + I_{(m-2)n} + \cdots + I_n)$.

Proof. We have

$$I_{mn} = b^{(m-1)n} I_n + b^{(m-2)n} I_n + \cdots + b^n I_n + I_n$$
$$= I_n (b^{(m-1)n} + b^{(m-2)n} + \cdots + b^n + 1),$$

and hence

$$I_{mn} - m I_n = I_n (b^{(m-1)n} - 1 + b^{(m-2)n} - 1 + \cdots + b^n - 1)$$
$$= I_n ((b-1) I_{(m-1)n} + (b-1) I_{(m-2)n} + \cdots + (b-1) I_n)$$
$$= I_n (b-1)(I_{(m-1)n} + I_{(m-2)n} + \cdots + I_n).$$

PROPOSITION 4. *If* $(n, b) = 1$, *then* $n \mid I_k$ *if and only if* $c(n) \mid k$.

Proof. By the division algorithm, there exist nonnegative integers j and r such that $k - jc(n) = r$, $r < c(n)$. Using Proposition 3(i), we obtain

$$I_k = b^{k-1} + \cdots + b^{jc(n)} + I_{jc(n)}$$
$$= b^{jc(n)} I_r + I_{c(n)} (b^{(j-1)c(n)} + \cdots + b^{c(n)} + 1).$$

If $c(n) \mid k$, then $r = 0$ and the above equation shows that since, by definition, $n \mid I_{c(n)}$, we must also have $n \mid I_k$. Conversely, if $n \mid I_k$, then the above equation shows that since $(n, b) = 1$, we must have $n \mid I_r$. However this would contradict the minimality of $c(n)$ unless $r = 0$, i.e., unless $c(n) \mid k$. The proposition is proved.

PROPOSITION 5. *If* $(m, b) = (n, b) = (m, n) = 1$, *then* $c(mn) = [c(m), c(n)]$, *the least common multiple of* $c(m)$ *and* $c(n)$.

Proof. Since $(m, n) = 1$, $m \mid I_k$ and $n \mid I_k$ if and only if $mn \mid I_k$. By Proposition 4, $m \mid I_k$ if and only if k is a multiple of $c(m)$ and $n \mid I_k$ if and only if k is a multiple of $c(n)$. Thus $mn \mid I_k$ if and only if k is a common multiple of $c(m)$ and $c(n)$. By definition, $c(mn)$ is then the least common multiple of $c(m)$ and $c(n)$.

PROPOSITION 6. *If* $n \mid (b-1)$, *then* $c(n) = n$.

Proof. Using Proposition 3(ii), we obtain

$$I_k - k = I_k - k I_1 = I_1 (b-1)(I_{k-1} + I_{k-2} + \cdots + I_1).$$

Since $n \mid (b-1)$, we deduce from the above equation that $n \mid I_k$ if and only if $n \mid k$. Now we take $k = c(n)$. By definition $n \mid I_{c(n)}$, and hence $n \mid c(n)$. Thus we have $c(n) \geqq n$. By Proposition 1, $c(n) \leqq n$, and therefore $c(n) = n$.

PROPOSITION 7. *Let* j *be a positive integer and* p *a prime number with* $(p, b) = 1$. *If* $c(p^j) = r$, *then either* $c(p^{j+1}) = r$ *or* $c(p^{j+1}) = pr$.

Proof. Since $c(p^j) = r$, we know $p^j \mid I_r$ but for $1 \leqq k < r$, $p^j \nmid I_k$. If $p^{j+1} \mid I_r$, then clearly $c(p^{j+1}) = r$. Thus, it suffices to show that if $p^{j+1} \nmid I_r$, then $c(p^{j+1}) = pr$. We first remark that if $p^{j+1} \mid I_n$, then certainly $p^j \mid I_n$. Hence by Proposition 4, n must be a multiple of $c(p^j) = r$, and therefore $c(p^{j+1})$ must be a multiple of r. Again by Proposition 4, we know $p^j \mid I_{ir}$ for every positive integer i, and hence $p^j \mid (I_{(k-1)r} + I_{(k-2)r} + \cdots + I_r)$ for each positive integer $k > 1$. Now, using Proposition 3(ii), we obtain, for $k > 1$ and $j \geqq 1$,

$$I_{kr} - k I_r = I_r (b-1)(I_{(k-1)r} + I_{(k-2)r} + \cdots + I_r)$$
$$\equiv 0 \pmod{p^{2j}}$$
$$\equiv 0 \pmod{p^{j+1}}.$$

In other words, $p^{j+1} | I_{kr}$ if and only if $p^{j+1} | k I_r$. Now $p^j | I_r$, and if $p^{j+1} \nmid I_r$, then we have just shown that $p^{j+1} | I_{kr}$ if and only if $p | k$. Thus $p^{j+1} | I_{pr}$ but $p^{j+1} \nmid I_{kr}$ if $1 \leq k < p$, i.e., $c(p^{j+1}) = pr$.

PROPOSITION 8. *Suppose p is a prime divisor of* $b-1$.

(i) *If p is odd, then* $c(p^2) = p^2$.

(ii) *If* $b \equiv 1 \pmod 4$, *then* $c(4) = 4$.

(iii) *If* $b \equiv 3 \pmod 4$, *then* $c(4) = 2$.

Proof. If $p | (b-1)$, then $b \equiv 1 \pmod p$, and we have

$$I_k = b^{k-1} + b^{k-2} + \cdots + b + 1 \equiv k \pmod p,$$

for every positive integer k. If, in addition, p is odd, then we have

$$\begin{aligned} I_{p-1} + I_{p-2} + \cdots + I_1 &\equiv (p-1) + (p-2) + \cdots + 1 \\ &= p(p-1)/2 \equiv 0 \pmod p. \end{aligned}$$

Thus, using Proposition 3(ii), we obtain

$$I_p - p = I_p - p I_1 = I_1 (b-1)(I_{p-1} + I_{p-2} + \cdots + I_1) \equiv 0 \pmod{p^2}.$$

We conclude $p^2 \nmid I_p$, i.e., $c(p^2) \neq p$. By Proposition 6, we know $c(p) = p$. Then by Proposition 7, we have either $c(p^2) = p$ or $c(p^2) = p^2$. Since the former has been ruled out, we must have $c(p^2) = p^2$.

For the case $p = 2$, we know, by Proposition 6, that if b is odd, $c(2) = 2$. Then by Proposition 7, we have $c(4) = 4$ or 2. Now if $b \equiv 1 \pmod 4$, then

$$I_2 = b + 1 \equiv 2 \pmod 4,$$

and thus $4 \nmid I_2$, i.e., $c(4) = 4$. On the other hand, if $b \equiv 3 \pmod 4$, then

$$I_2 \equiv 3 + 1 \equiv 0 \pmod 4.$$

Thus $4 | I_2$ and $c(4) = 2$.

PROPOSITION 9. *Let j be a positive integer and p a prime number with* $(p, b) = 1$. *If* $c(p^j) = r$ *and* $c(p^{j+1}) = pr$, *then* $c(p^{j+2}) = p^2 r$.

Proof. By Proposition 3(i), we have, for each positive integer k,

$$I_{kr} = I_r (b^{(k-1)r} + b^{(k-2)r} + \cdots + b^r + 1).$$

Using Proposition 3(ii) and then applying the above for $k = 1, \ldots, p-1$, we obtain

$$\begin{aligned} I_{pr} - p I_r &= I_r (b-1)(I_{(p-1)r} + I_{(p-2)r} + \cdots + I_r) \\ &= I_r^2 (b-1) B, \end{aligned}$$

where $B = (b^{(p-2)r} + \cdots + b^r + 1) + \cdots + (b^r + 1) + 1$. If $(p, b-1) = 1$, then by Proposition 2(i), we have $b^r \equiv 1 \pmod p$. Since, by hypothesis, we also have $(p, b) = 1$, we know that in this case $p \neq 2$. Hence, if $p \nmid (b-1)$, then

$$B \equiv (p-1) + (p-2) + \cdots + 2 + 1 = p(p-1)/2 \equiv 0 \pmod p.$$

Thus either $p | (b-1)$ or $p | B$, i.e., it is always true that $p | (b-1)B$. By hypothesis, $c(p^j) = r$, so that $p^j | I_r$. Therefore, we have

$$\begin{aligned} I_{pr} - p I_r = I_r^2 (b-1) B &\equiv 0 \pmod{p^{2j+1}} \\ &\equiv 0 \pmod{p^{j+2}}. \end{aligned}$$

In other words, $p^{j+2}|I_{pr}$ if and only if $p^{j+2}|pI_r$. But, by hypothesis, $c(p^{j+1})=pr$, so that $p^{j+1} \nmid I_r$ and thus $p^{j+2} \nmid pI_r$. Hence $p^{j+2} \nmid I_{pr}$, i.e., $c(p^{j+2}) \neq pr$. By Proposition 7, the only remaining alternative is $c(p^{j+2})=p^2 r$.

PROPOSITION 10. *Let p be a prime number with $(p, b)=1$.*
(i) *If p is odd and $p \mid (b-1)$, then $c(p^j)=p^j$ for every positive integer j.*
(ii) *If $p \nmid (b-1)$, then $c(p^j)<p^j$ for every positive integer j.*
(iii) *If $b \equiv 1 \pmod 4$, then $c(2^j)=2^j$ for every positive integer j.*
(iv) *If $b \equiv 3 \pmod 4$, then $c(2)=2$ but $c(2^j)<2^j$ for every integer $j \geqq 2$.*

Proof. To prove (i) and (iii), we note that if $p \mid (b-1)$ and either p is odd or $b \equiv 1 \pmod 4$, then, by Propositions 6 and 8, we have $c(p)=p$ and $c(p^2)=p^2$. By Proposition 9, we see that if $c(p^j)=p^j$ and $c(p^{j+1})=p^{j+1}$, then $c(p^{j+2})=p^{j+2}$. By the principle of mathematical induction, the proof is complete.

To prove (ii) and (iv), we again proceed by induction. By Proposition 7, if $c(p^j)<p^j$, then $c(p^{j+1}) \leqq pc(p^j)<p^{j+1}$. By Proposition 2(ii), if $p \nmid (b-1)$, then $c(p)<p$, and by Proposition 8(iii), if $b \equiv 3 \pmod 4$, then $c(4)<4$. By induction, the proof of the proposition is complete.

PROPOSITION 11. *Let n be a positive integer with $(n, b)=1$.*
(i) *If $b \not\equiv 3 \pmod 4$, then $c(n)=n$ if and only if every prime factor of n is also a factor of $b-1$.*
(ii) *If $b \equiv 3 \pmod 4$, then $c(n)=n$ if and only if $4 \nmid n$ and every prime factor of n is also a factor of $b-1$.*

Proof. Let $n=p_1^{a_1}\cdots p_r^{a_r}$ be the unique factorization of n into prime factors, $p_i \neq p_j$ if $i \neq j$. Then by applying Proposition 5 repeatedly, we obtain

$$\begin{aligned} c(n) &= [c(p_1^{a_1}), \ldots, c(p_r^{a_r})] \\ &\leqq p_1^{a_1} \ldots p_r^{a_r} = n \end{aligned}$$

with equality holding if and only if $c(p^a)=p^a$ whenever $p^a \mid n$. The conclusions of this proposition are now easily deduced by applying Proposition 10.

PROPOSITION 12. *If $(n, b)=1$, then $c(n)=n-1$ if and only if n is a prime which does not divide $b-1$, and b is a primitive root of n.*

Proof. Suppose $c(n)=n-1$. First we show that n cannot have more than one prime factor. Suppose $n=jk$ where $j>1$, $k>1$, and $(j, k)=1$. We cannot have both $c(j)=j$ and $c(k)=k$, for in this case, we get $c(n)=[c(j), c(k)]=jk=n$. We may therefore assume that $c(j)<j$. But then

$$c(n) \leqq c(j)c(k) \leqq (j-1)k=n-k<n-1.$$

Hence n must be a power of a prime p. Assume $n=p^j$. By Proposition 10, $c(n)<n$ only if $p \nmid (b-1)$ or if $p=2$ and $b \equiv 3 \pmod 4$. In the first case, $c(p) \leqq p-1$ by Proposition 2(ii), and in the second case $c(4)=2<4-1$ by Proposition 8(iii). Now if $c(p^j) \leqq p^j-1$ for some positive integer j, then by Proposition 7, $c(p^{j+1}) \leqq pc(p^j) \leqq p^{j+1}-p<p^{j+1}-1$. By the principle of mathematical induction, we conclude that $c(p^j)<p^j-1$ if $j \geqq 2$. Thus $c(n)=n-1$ only if n is a prime p which does not divide $b-1$.

The proof of the theorem will be complete if we can show that whenever $(p, b-1)=(p, b)=1$, then $c(p)=p-1$ if and only if b is a primitive root of p. Now, according to Proposition 2(i), $c(p)$ is the smallest positive integer k such that $b^k \equiv 1 \pmod p$. If b is a primitive root of p, then that smallest positive integer is $p-1$ and otherwise it is a proper divisor of $p-1$. The proof is complete.

Reference

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NOTES

The Number of Regions Determined by a Convex Polygon

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The purpose of this note is to give a short proof of the following well-known problem discussed in the delightful book, *Mathematical Gems* [**2**, p. 99]: *Draw all lines joining the vertices of a convex n-gon, C, in the Euclidean plane. Let C have the property that no three lines meet in an interior point. What is the number N of interior regions of C?*

By moving a line across the n-gon we will show that

$$N = \binom{n}{4} + \binom{n-1}{2}. \tag{1}$$

This method also allows us to give the number of regions determined by an arbitrary convex n-gon. Further, it gives an immediate solution to a related well-known problem [**1**, **3**] in Euclidean N-dimensional space which we consider at the end of this note.

To establish (1) we first observe that since there are only a finite number of edges and diagonals, we may choose a line, c, which is not parallel to any of the lines of the figure. Call such a line a **counting line.** Choose a counting line, c, such that the n vertices of the figure C are in one of the half-planes determined by c. On translating c (hereafter called moving c) through C the counting line passes through each vertex one at a time. This gives the vertices an order, $v_0, v_1, v_2, \ldots, v_{n-1}$, where $i < j$ implies that c passes through v_i before v_j when c moves through C.

Consider now a 4-subset of the n vertices. The two diagonals of this quadrangle determine exactly one interior point. Conversely, since no three diagonals of C meet in an interior point, the two diagonals which do meet in a given interior point determine a 4-subset of the n vertices. Hence each 4-subset of the n vertices determines exactly one interior point and each interior point determines a unique 4-subset of the vertices. The number of interior points is then $\binom{n}{4}$. (This fact is proved in [**2**, p. 101].)

We count the total number of regions of C by counting a given region when the counting line first intersects it as c moves through the figure. A region is first entered by c by passing through either an interior point or a vertex point.

Consider an interior point P. Before c passes through P, c intersects three of the four regions having P as a vertex. These three regions have thus been counted earlier. As c passes through P exactly one new region is entered and is counted at this time. Hence when c passes through an interior point, the count increases by one.

Consider now a vertex point. At each vertex there are $n-2$ triangular regions. However when c reaches a particular vertex it has already entered some of these. Whence there are fewer than $n-2$ new regions by which the count should increase. To obtain the precise count consider the vertex v_j. Each of the j segments, $v_jv_0, v_jv_1, \ldots, v_jv_{j-1}$, intersects the trailing half-plane of c and may be associated with one of the $n-2$ triangles at v_j. These have been counted already. Consequently c enters exactly $n-2-j$ new regions at v_j (see FIGURE 1). Hence the count increases by exactly $n-2-j$ when c passes through the vertex point v_j. We note that when c passes through v_{n-1} it no longer intersects C.

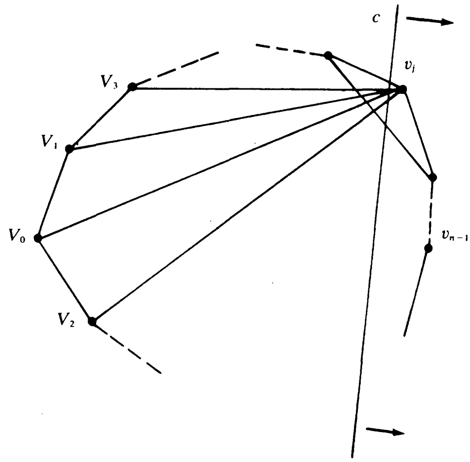

FIGURE 1

By moving c through C, we pass through each region exactly once. Since we obtain one region for each interior point and $n-2-j$ regions for the vertex v_j we obtain (1):

$$N = \binom{n}{4} + (n-2)+(n-3)+\cdots+1 = \binom{n}{4} + \binom{n-1}{2}.$$

Now let C be any convex n-gon. Draw all lines joining its vertices; in this case any number of them may meet in a common point. Call an intersection point P an **i-point** if exactly i lines of the figure pass through P, hence $2 \leqq i \leqq [n/2]$. Let T_i be the number of i-points determined by the figure.

Consider the $2i$ regions determined by an i-point, P. Before a counting line c passes through P the i-lines divide c into $i+1$ segments. The counting line thus intersects $i+1$ regions already. When c passes through P it enters $2i-(i+1)=i-1$ new regions. Hence, counting the regions as above, when c passes through an i-point the count increases by exactly $i-1$. The count at the vertex v_j is unchanged and is $n-2-j$. Hence the total number of regions in C is

$$\binom{n-1}{2} + \sum_{i=2}^{[n/2]} (i-1)T_i, \quad \text{where} \quad \binom{n}{4} = \sum_{i=2}^{[n/2]} \binom{i}{2} T_i.$$

This last sum is obtained by noticing that the number of distinct 4-subsets of vertices determined by an i-point is $\binom{i}{2}$.

A counting line may also be used to count the number M of regions determined by n lines in the Euclidean plane E_2. Let S be a collection of n pairwise nonparallel lines in E_2. Let W_i be the number of i-points determined by S, $2 \leqq i \leqq n$.

We may place a counting line c such that all points of intersection are in one half-plane. The n lines of S meet c in n distinct points. Counting a region when it is first entered by c, the placement of c counts $n+1=\binom{n}{1}+\binom{n}{0}$ regions. As c passes through an i-point, the count increases by $i-1$. Thus, $M=\binom{n}{1}+\binom{n}{0}+\Sigma_{i=2}^{n}(i-1)W_i$ where $\binom{n}{2}=\Sigma_{i=2}^{n}\binom{i}{2}W_i$.

Ben Manvel (private communication) has pointed out that this method easily counts spaces formed in an analogous manner in Euclidean N-dimensional space E_N if one restriction is met. Let $M(N,n)$ be the number of spaces determined by a collection S of n hyperplanes of E_N such that each set of N

hyperplanes determines a unique affine point (vertex) and each vertex point determines a unique set of N hyperplanes.

As above, we may place a counting hyperplane c such that all of the $\binom{n}{N}$ vertex points are in one half-space. Since each vertex determines a unique N-set of hyperplanes of S, the hyperplanes section the hyperplane c into n $(N-2)$-flats of c (hyperplanes of c) such that each intersection point in c determines a unique set of $N-1$ hyperplanes of c. Counting a space when it is first entered by c, the placement of c then counts $M(N-1, n)$ spaces. Moving c through $\binom{n}{N}$ vertices we have

$$M(N, n) = M(N-1, n) + \binom{n}{N} = \sum_{i=0}^{N} \binom{n}{i}. \tag{5.1}$$

If $n < N$, then no vertex points are determined. However, E_N is divided into 2^n spaces to which (5.1) reduces if $\binom{n}{i}$ is said to be zero when $n < i$.

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A New Proof of Routh's Theorem

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We offer here a simple proof of Routh's theorem, a classic result of geometry: *If the sides BC, CA, AB of a triangle are divided at points D, E, F in the ratios* 1 *to r*, 1 *to s*, 1 *to t, respectively, then the area of the triangle formed by AD, BE, CF is*

$$\frac{(rst-1)^2}{(rs+s+1)(st+t+1)(tr+r+1)} \tag{1}$$

times that of triangle ABC. A special case of this is Ceva's theorem, that *the lines AD, BE, CF are concurrent if and only if rst* = 1; the lines AD, BE, CF are sometimes called "Cevians." This theorem in turn implies that the medians of a triangle are concurrent lines, as also are the altitudes, and the angle bisectors. There are many proofs available of the Routh result, for example Coxeter [**1**, p. 211, pp. 216–220]; Kay [**2**, pp. 205–207], and Melzak [**3**, pp. 7–9]. The proof given here, besides being brief, makes use only of very elementary coordinate geometry.

To begin the proof of Routh's theorem, we write a, b, c, d, e, f, z for the areas of the seven regions into which the triangle is divided, as in FIGURE 1. The unit of length may be chosen so that triangle ABC has area 1, and so we prove that z equals the expression (1).

First we prove that if P is the intersection of AD and BE, then $AD = (rs + s + 1)PD$. Impose a coordinate system with the x-axis along BC, so that the coordinates of B, C, A may be taken as (0, 0), $(u, 0)$, (v, w). We use the well-known result that if (x, y) divides a line segment from (x_1, y_1) to (x_2, y_2)

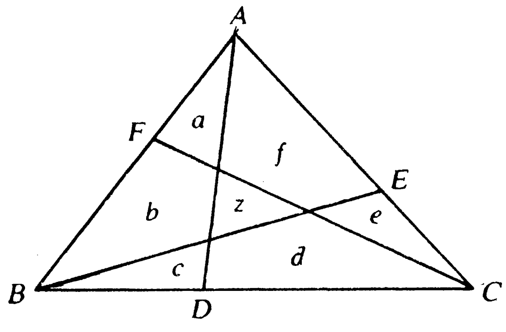

FIGURE 1

in the ratio p to q, then x is the weighted average $(qx_1 + px_2)/(p + q)$, and similarly for y. Thus D and E have coordinates

$$\left(\frac{u}{1+r}, 0\right) \quad \text{and} \quad \left(\frac{us+v}{1+s}, \frac{w}{1+s}\right).$$

If we denote AP/PD and PE/BP by λ and θ, the coordinates of P can be expressed as a point on AD, and as a point on BE. Equating the coordinates of P so obtained we have

$$\frac{v+(\lambda u/(1+r))}{1+\lambda} = \frac{us+v}{(1+s)(1+\theta)}, \qquad \frac{w}{1+\lambda} = \frac{w}{(1+s)(1+\theta)}.$$

Dividing these equations we get $\lambda = rs + s$, as was to be proved.

Now the areas of triangles BPD and ABD have the same ratio as PD to AD, and so we have $c = (a + b + c)/(rs + s + 1)$. Similarly the areas of ABD and ABC have the same ratios as BD to BC, and so, since the area of ABC is 1,

$$(2) \qquad a+b+c = \frac{1}{r+1}, \qquad c = \frac{1}{(r+1)(rs+s+1)}.$$

If we add this formula for c to the analogues for a and e, we get

$$(3) \qquad a+c+e = \frac{1}{(r+1)(rs+s+1)} + \frac{1}{(s+1)(st+t+1)} + \frac{1}{(t+1)(tr+r+1)}.$$

If we add the first of equations (2) to the analogues for $c + d + e$ and $e + f + a$ we get

$$2(a+c+e)+(b+d+f) = \frac{1}{r+1} + \frac{1}{s+1} + \frac{1}{t+1}.$$

The subtraction of (3) from this gives

$$(a+c+e)+(b+d+f) = \frac{s}{rs+s+1} + \frac{t}{st+t+1} + \frac{r}{tr+r+1}.$$

But the area of ABC is 1, and so

$$z = 1-(a+c+e)-(b+d+f) = 1 - \frac{s}{rs+s+1} - \frac{t}{st+t+1} - \frac{r}{tr+r+1},$$

and this is equivalent to (1) by elementary algebra.

Because of the reliance on a diagram, the proof is not complete. If the point D is close to C, then P, the intersection of AD and BE, would be inside the triangle ACF, and the argument above does not apply to such a diagram. However, if P is inside the triangle ACF, we consider the mirror image $A'B'C'$ of ABC. We can apply formula (1), suitably modified, to $A'B'C'$. Using the counterclockwise direction around this triangle, we note that the sides $A'C'$, $C'B'$, $B'A'$ are divided in the ratios 1 to $1/s$, 1 to $1/r$, 1 to $1/t$. Thus we replace r, s, t in formula (1) by $1/s$, $1/r$, $1/t$, only to find that (1) is invariant under this transformation. The proof of Routh's theorem for any positive real numbers r, s, t is now complete.

For a more general proof that extends the result to permit both internal and external divisions of the sides of ABC, see Coxeter [1, pp. 216–220]. The general result requires absolute value signs on formula (1). More than that, in order to allow for any choices of D, E, F on the sides BC, CA, AB including the vertices themselves, the ratio numbers r, s, t should be in homogeneous form. The general formulation of the Routh theorem would then be: if the sides BC, CA, AB of a triangle are divided at points D, E, F in the ratios p_1 to q_1, p_2 to q_2, p_3 to q_3, respectively, then the area of the triangle formed by AD, BE, CF is

$$\frac{(p_1p_2p_3 - q_1q_2q_3)^2}{|(p_1p_2 + q_1q_2 + p_1q_2)(p_2p_3 + q_2q_3 + p_2q_3)(p_1p_3 + q_1q_3 + p_3q_1)|} \tag{4}$$

times the area of the triangle ABC. This formula is valid for any location of the points D, E, F on the sides of the triangle, except those that produce a zero denominator. The denominator of formula (4) is zero if and only if two or more of the Cevian lines AD, BE, CF are parallel. The expression (4) is given here because this general formulation is not often given in treatments of this topic.

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k-Transposable Integers

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Let k be a positive integer. Then the positive integer x is called **k-transposable** if and only if it has the property that when its leftmost digit is moved into the units place, the resulting integer is equal to kx. It is immediately apparent that x is 1-transposable if and only if all of its digits are identical, so we assume henceforth that $k > 1$. Letting t be a nonnegative integer, we introduce the notation $\bar{N}_t$ to mean the following:

(i) if $t > 1$, $\bar{N}_t$ is the integer obtained by concatenating N t times,

(ii) if $t = 1$, $\bar{N}_1$ is just the integer N,

(iii) if $t = 0$, $\bar{N}_0$ is the statement that N is entirely absent.

In this paper, it will be shown that *for $k > 1$, k-transposable integers only occur when $k = 3$, and that the only 3-transposable integers are $\overline{142857}_m$ and $\overline{285714}_m$, where $m \geqq 1$.*

Let x have n digits, $n \geqq 2$, and label the digits of x in the following manner: call x_0 the units digit; x_1, the tens digit; and continue until x_{n-1}, the leftmost digit of x. Then we may note these elementary facts at the outset:

(a) $0 \leqq x_i \leqq 9$ for $0 \leqq i < n-1$,

(b) $0 < x_{n-1} \leqq 9$,

(c) $k \leqq 9$, since $k \geqq 10$ would imply that kx has at least one more digit than x, a contradiction.

By definition, x is k-transposable if and only if the following equation is satisfied: $(\Sigma_{i=0}^{n-2} 10^{i+1}x_i) + x_{n-1} = k(\Sigma_{i=0}^{n-1} 10^i x_i)$. Rearranging terms, we obtain:

$$(1) \qquad (10^{n-1}k - 1)x_{n-1} = (10 - k)\left(\sum_{i=0}^{n-2} 10^i x_i\right).$$

Let d be the greatest common divisor of $10^{n-1}k - 1$ and $10 - k$. Then $d|(10-k)$ implies $d|10^{n-1}(10-k)$, i.e., $d|(10^n - 10^{n-1}k)$. Since $d|(10^{n-1}k - 1)$, $d|(10^n - 10^{n-1}k) + (10^{n-1}k - 1))$, or $d|(10^n - 1)$. Now $10^n - 1 = 999\ldots9$, where the digit 9 appears n times. Since $2 \leqq k \leqq 9$ and $d|(10-k)$, it follows that $1 \leqq d \leqq 8$. But $d|(10^n - 1)$, so $d = 1, 3$, or 7.

Case 1: $d = 1$. In this case, $10^{n-1}k - 1$ and $10 - k$ are relatively prime, so it follows from (1) that $10^{n-1}k - 1$ must divide $\Sigma_{i=0}^{n-2} 10^i x_i$. But since $2 \leqq k \leqq 9$, $10^{n-1}k - 1$ is an n-digit number while $\Sigma_{i=0}^{n-2} 10^i x_i$ has at most $n-1$ digits, making the aforesaid division impossible. Thus, no k-transposable integer exists for $d = 1$.

Case 2: $d = 3$. Since $d|10-k$, the subcases $k = 4$ and $k = 7$ must be considered. When $k = 4$, (1) becomes $(4(10^{n-1}) - 1)x_{n-1} = 6(\Sigma_{i=0}^{n-2} 10^i x_i)$, which reduces to $1333\ldots3x_{n-1} = 2(\Sigma_{i=0}^{n-2} 10^i x_i)$, where the digit 3 appears $n-1$ times. But $x_{n-1} \neq 1$, since if it were, 2 would have to divide $1333\ldots3$. Hence, the left side of this last equation is greater than $2(10^{n-1})$, while the right side is necessarily less than $2(10^{n-1})$. Thus, $k \neq 4$. When $k = 7$, (1) becomes $(7(10^{n-1}) - 1)x_{n-1} = 3(\Sigma_{i=0}^{n-2} 10^i x_i)$, which reduces to $2333\ldots3x_{n-1} = \Sigma_{i=0}^{n-2} 10^i x_i$, where the digit 3 appears $n-1$ times. But this is an immediate contradiction, since the left side of this last equation is an integer with at least n digits, while the right side is one with at most $n-1$ digits. Thus, no k-transposable integer exists for $d = 3$.

Case 3: $d = 7$. Here $d|(10-k)$ implies that $k = 3$, and this, together with the fact that $d|(10^{n-1}k - 1)$, yields $7|(3(10^{n-1}) - 1)$, or $3(10^{n-1}) \equiv 1 \pmod 7$. But $3(10^{n-1}) \equiv 3^n \pmod 7$, so $3^n \equiv 1 \pmod 7$. Since 3 is a primitive root modulo 7, it follows that $n = 6m$, where m is a positive integer. Then (1) becomes $(3(10^{n-1}) - 1)x_{n-1} = 7(\Sigma_{i=0}^{n-2} 10^i x_i)$, and upon division by 7, we obtain:

$$(42857\overline{142857}_{m-1})x_{n-1} = \sum_{i=0}^{n-2} 10^i x_i.$$

Since the integer on the left can have at most $n-1$ digits, the only possible values of x_{n-1} are 1 and 2, which produce the 3-transposable integers $\overline{142857}_m$ and $\overline{285714}_m$ respectively.

Extensions of Some Geometric Inequalities

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For any triangle ABC, it is known [**1**, p. 12] that

$$(1) \qquad abc \geqq (a+b-c)(b+c-a)(c+a-b), \qquad \{E\},$$

where for convenience the symbol $\{E\}$ denotes "with equality if and only if the elements are all congruent." It is to be noted that we are only considering the class of nondegenerate triangles, otherwise, we could have equality for $a = b$, $c = 0$. The author was led to a generalization of (1) by means of the following geometric interpretation [**2**, pp. 3–4]: If one considers a related triangle $A'B'C'$ with sides

$$a' = \frac{b+c}{2}, \qquad b' = \frac{c+a}{2}, \qquad c' = \frac{a+b}{2},$$

then since $s = s'$ and $A'B'C'$ is "closer" to being an equilateral triangle than ABC, one would expect the area inequality $\Delta' \geqq \Delta$ and this is equivalent to (1). More generally, if

$$a = \sum w_i a_i, \qquad b = \sum w_i b_i, \qquad c = \sum w_i c_i$$

are the sides of a triangle where a_i, b_i, c_i are the respective sides of n given triangles $A_iB_iC_i$, $w_i \geqq 0$, $\Sigma w_i = 1$ and the index $i = 1, 2, \ldots, n$, then

$$\sqrt{\Delta} \geqq \sum w_i \sqrt{\Delta_i} \tag{2}$$

with equality iff the n given triangles are directly similar.

An analytic extension of (1), due to Mitrinović and Adamović [**3**, pp. 208–9] is that

$$a_1 a_2 \cdots a_n \geqq \Pi\{a_2 + a_3 + \cdots + a_n - (n-2)a_1\}, \qquad \{E\}, \tag{3}$$

where $a_i > 0$ and all the factors on the right-hand side are nonnegative.

Now we give an extension of (3) which is analogous to the extension (2) of (1). We shall show that

$$P^{1/n} \geqq \sum_{i=1} w_i P_i^{1/n} \tag{4}$$

where

$$P_i = \prod_{j=1}^{n} (T_i - \lambda a_{ij}), \qquad T_i = \sum_{j=1}^{n} a_{ij},$$

$$P = \prod_{j=1}^{n} (T - \lambda a_j), \quad T = \sum_{j=1}^{n} a_j,$$

$$a_j = \sum_{i=1}^{m} w_i a_{ij}, \qquad \sum_{i=1}^{m} w_i = 1,$$

$$a_{ij} > 0, \ T_i - \lambda a_{ij} \geqq 0 \quad \text{and} \quad w_i \geqq 0.$$

There is equality iff the m vectors $(a_{i1}, a_{i2}, \ldots, a_{in})$, $i = 1, 2, \ldots, m$ are parallel. The proof follows from Jensen's extension of Hölder's inequality [**3**, p. 52]. Since

$$T = \sum_{j=1}^{n} \sum_{i=1}^{m} w_i a_{ij},$$

$$P = \prod_{k=1}^{n} \sum_{i=1}^{m} w_i \left\{ \sum_{j=1}^{n} a_{ij} - \lambda a_{ik} \right\}.$$

Then,

$$P^{1/n} \geqq \sum_{i=1}^{m} w_i \prod_{k=1}^{n} \{T_i - \lambda a_{ik}\}^{1/n}.$$

In particular, if $a_{ij} = b_{i+j-1}$ where $b_{j+n} = b_j$ for all i and j, then (4) reduces to

$$\prod_{j=1}^{n} (S' - \lambda b'_j) \geqq \prod_{j=1}^{n} (S - \lambda b_j) \tag{5}$$

where

$$b'_j = \sum_{i=1}^{m} w_i b_{i+j-1}, \qquad S' = \sum_{j=1}^{n} b'_j = \sum_{j=1}^{n} b_j = S,$$

and with equality if $b_1 = b_2 = \cdots = b_n$.

Inequality (3) is now a special case of (5) corresponding to $m = n$, $\lambda = n-1$, $w_1 = 0$ and $w_2 = w_3 = \cdots = w_n = 1/(n-1)$. Also, by letting $m = n = 4$, $\lambda = 2$, $w_i = \frac{1}{4}$, (5) reduces to the isoperimetric inequality for cyclic quadrilaterals.

Finally, by letting $S - \lambda b_j = u_j$, $U = \Sigma_{j=1}^n u_j$, (5) can be written in the dual form

$$\prod_{j=1}^{n} \sum_{i=1}^{m} w_i u_{i+j-1} \geqq u_1 u_2 \cdots u_n.$$

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Integer Representations and Complete Sequences

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For $\{s_i\}_1^\infty$ a strictly increasing sequence of positive integers, Lee [1] has defined $\{s_i\}$ to be a **Scherk sequence** if every positive integer n satisfying $s_m < n \leqq s_{m+1}$ has a representation in at least one of the two forms:

(A) $n = s_m \pm s_{m-1} \pm \cdots \pm s_1$

(B) $n = 2s_m \pm s_{m-1} \pm \cdots \pm s_1$,

where the signs in either representation are appropriately chosen. This definition was motivated by Scherk's original conjecture [2] relating to the representation of primes in the given form when $\{s_i\}$ is the sequence of positive primes with unity adjoined. Lee [1] then characterized the set of Scherk sequences (having $s_1 = 1$) by the following necessary and sufficient conditions:

(1) whenever s_t is even, then $\Sigma_1^t s_i$ is odd and $s_{t+1} = s_t + 1$.

(2) $s_{m+1} \leqq 1 + \Sigma_1^m s_i$ for $m = 1, 2, 3, \ldots$.

Now, an arbitrary sequence of positive integers $\{s_i\}$ is said to be **complete** if every positive integer n can be expressed as a sum of distinct terms from the sequence. If we assume additionally that $\{s_i\}$ is nondecreasing and $s_1 = 1$, then a necessary and sufficient condition for completeness [3] is

(3) $s_{m+1} \leqq 1 + \Sigma_1^m s_i$ for all $m \geqq 1$.

Henceforth, we assume $\{s_i\}$ is an arbitrary nondecreasing sequence of positive integers with $s_1 = 1$, so that completeness is equivalent to the inequality condition (3). In particular Lee's condition (2) requires that a Scherk sequence be complete.

The purpose of the present note is to show that, for given $\{s_i\}$, every positive integer n satisfying $s_m < n \leqq s_{m+1}$ has a representation (with proper choices of signs) in exactly one of the forms

(C) $n = s_m \pm s_{m-1} \pm \cdots \pm s_1$

(D) $n = s_m \pm s_{m-1} \pm \cdots \pm s_1 + 1$

if and only if the sequence $\{s_i\}$ is complete.

The difference between the (A)-(B) and (C)-(D) representations resides in the treatment of the case where n and $\Sigma_1^m s_i$ have different parity. In (B), following Scherk, the first term s_m is doubled while in (D) unity is added when needed to give the required parity correspondence. The latter expedient seems more natural, and still permits [using only condition (2)] representations which retain the basic Scherk property that all terms from $\{s_i\}$ which are strictly less than n must be used (with appropriate sign choices) in the representation for the positive integer n.

We begin our proof by restating in Lemma 1 a result from [**3**; p. 557] and using it to prove a second lemma which will lead directly to our main result.

LEMMA 1. *Let* $\{s_i\}$ *be a nondecreasing sequence of positive integers with* $s_1 = 1$ *and satisfying* $s_{m+1} \leqq 1 + \Sigma_1^m s_i$ *for all* $m \geqq 1$. *Then for any integer n satisfying* $0 \leqq n < 1 + \Sigma_1^k s_i$, *there exist binary coefficients* $\alpha_i = \alpha_i(n)$ *such that* $n = \Sigma_1^k \alpha_i s_i$ ($\alpha_i = 0$ *or* 1 *for each* i).

LEMMA 2. *Let* $\{s_i\}_1^\infty$ *be a complete sequence of positive integers. If n is a positive integer satisfying*

(i) $s_k \leqq n \leqq s_{k+1}$

(ii) $n \equiv \Sigma_1^k s_i \pmod 2$,

then n has a representation

$$n = s_k + \sum_1^{k-1} \varepsilon_i s_i,$$

where each $\varepsilon_i = \varepsilon_i(n)$ *is either* $+1$ *or* -1.

Proof. By completeness of $\{s_i\}$, $s_k \leqq n \leqq s_{k+1}$ implies $s_k \leqq n \leqq 1 + \Sigma_1^k s_i$, or, *a fortiori*, $0 \leqq n - s_k \leqq 2 + \Sigma_1^{k-1} s_i$.

Adding $\Sigma_1^{k-1} s_i$ to both sides of the inequality and multiplying by 1/2, we obtain

$$0 \leqq \frac{1}{2}\left(n - s_k + \sum_1^{k-1} s_i\right) < 1 + \sum_1^{k-1} s_i.$$

By Lemma 1, noting that (ii) implies $\frac{1}{2}(n - s_k + \Sigma_1^{k-1} s_i)$ is an integer, there exist binary coefficients α_i such that

$$\frac{1}{2}\left(n - s_k + \sum_1^{k-1} s_i\right) = \sum_1^{k-1} \alpha_i s_i$$

or

$$n = s_k + \sum_1^{k-1} (2\alpha_i - 1)s_i.$$

Noting that $2\alpha_i - 1 \equiv \varepsilon_i$ is always ± 1 completes the proof.

THEOREM 1. *Let* $\{s_i\}_1^\infty$ *be a nondecreasing complete sequence of positive integers, with* $s_1 = 1$. *Then for each positive integer n satisfying* $s_k < n \leqq s_{k+1}$ ($k \geqq 1$), *we have either*

(C) $n = s_k \pm s_{k-1} \pm \cdots \pm s_1$

or

(D) $n = s_k \pm s_{k-1} \pm \cdots \pm s_1 + 1$,

for an appropriate selection of signs. [*Clearly* (C) *applies when* $n \equiv \Sigma_1^k s_i \pmod 2$ *while* (D) *holds in the remaining case,* $n \equiv 1 + \Sigma_1^k s_i \pmod 2$].

Proof. Since both n and $n - 1$ satisfy condition (i) of Lemma 2 and exactly one of them satisfies (ii) of Lemma 2, the theorem follows immediately.

COROLLARY 1. *Let* $\{p_i\}_1^\infty$ *with* $p_1 = 1$ *be the sequence of positive primes arranged in increasing order. Then every positive integer n satisfying* $p_k < n \leqq p_{k+1}$ *can be written in one of the two forms*

(C′) $n = p_k \pm p_{k-1} \pm \cdots \pm p_1$

(D′) $n = p_k \pm p_{k-1} \pm \cdots \pm p_1 + 1$ *with suitable choice of signs.*

This follows directly upon recalling that the sequence of positive primes (with unity adjoined) is a complete sequence (implied by Bertrand's postulate and induction). As a special case, we have

$$p_{k+1} = p_k \pm p_{k-1} \pm \cdots \pm p_1 \text{ for } k \text{ even and } \geqq 2$$

$$p_{k+1} = p_k \pm p_{k-1} \pm \cdots \pm p_1 + 1 \text{ for } k \text{ odd and } \geqq 1,$$

which shows that each prime can be written as a linear combination (with appropriate ± 1 coefficients) of all preceding primes (plus a unity term when needed from parity considerations).

THEOREM 2. *Let* $\{s_i\}_1^\infty$ *be an arbitrary nondecreasing sequence of positive integers with* $s_1 = 1$. *Every positive integer n satisfying* $s_k < n \leqq s_{k+1}$ *has a representation in at least one of the forms* (C) *or* (D), *if and only if* $\{s_i\}$ *is complete.*

Proof. If every positive integer in $(s_k, s_{k+1}]$ has such a representation, then in particular, for each $k \geqq 1$,

$$s_{k+1} = s_k \pm s_{k-1} \pm \cdots \pm s_1 + \gamma_k,$$

where γ_k is either 0 or 1 depending on which of the two representations applies. Then

$$s_{k+1} \leqq s_k + s_{k-1} + \cdots + s_1 + 1 = 1 + \sum_1^k s_i, \qquad (k \geqq 1)$$

which implies completeness. The converse is a restatement of Theorem 1.

Note that the class of Scherk sequences is a proper subset of the set of complete sequences; for example, the set of Fibonacci numbers ($F_1 = 1$, $F_2 = 2$, $F_{n+2} = F_n + F_{n-1}$ for $n \geqq 1$) is complete but, as shown in Lee's paper, is not a Scherk sequence. The expansion considered here appears to be more natural than the Scherk expansion while retaining much of its flavor, namely the expansion of each integer n satisfying $s_k < n \leqq s_{k+1}$ as a linear combination, with $\pm$ coefficients, of *all* the terms $s_1, \ldots, s_k$ strictly less than n, plus a parity correction term of $+1$ when needed.

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An Iff Fixed Point Criterion

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In [1], Bennett and Fisher give an easy proof of Edelstein's theorem [2] which states that any mapping g of a compact metric space (X, d) into itself satisfying the condition

(1) $$d(g(x), g(y)) < d(x, y), \quad x \neq y,$$

has a unique fixed point. In this note we show that a judicious modification of (1) yields a necessary and sufficient condition for the existence of a fixed point. Our modification depends on the existence of a function that commutes with g. If f and g are mappings of a set X into itself, f and g are said to

commute if $f(g(x))=g(f(x))$ for all x in X. For example, the nth composition f^n of f, and the identity map ($i(x)=x$, $x\in X$) commute with f. With this definition in mind we can state and prove the following result:

THEOREM. *Let g be a continuous mapping of a compact metric space (X,d) into itself. Then g has a fixed point iff there is a mapping $f\colon X\to g(X)$ which commutes with g and satisfies*

$$d(f(x),f(y))<d(g(x),g(y)),\qquad g(x)\neq g(y). \tag{2}$$

Proof. (Necessity): By hypothesis there exists $a\in X$ such that $g(a)=a$. Define f by $f(x)=a$ ($x\in X$). Then $f(X)=\{a\}\subset g(X)$. Moreover, $f(g(x))=a$ for all x in X since $g(x)\in X$. But $g(f(x))=g(a)=a$ if $x\in X$. Thus $f(g(x))=g(f(x))$ for all $x\in X$; i.e., f and g commute. Finally, $g(x)\neq g(y)$ implies $d(g(x),g(y))>0=d(a,a)=d(f(x),f(y))$, so that condition (2) holds.

(Sufficiency): Since d is a metric for X, d is continuous; therefore, since g is continuous, the composite $d(g(x),g^2(x))$ is continuous on the compact space X. Consequently, this real valued composite attains a minimum value on X. In other words, there exists $a\in X$ such that

$$d(g(a),g^2(a))\leqq d(g(x),g^2(x)),\qquad x\in X. \tag{i}$$

We assert that $g(a)=g^2(a)$; i.e., $g(a)$ is a fixed point of g. If not, (2) and the commutativity of f and g yield

$$d(f(a),g(f(a)))=d(f(a),f(g(a)))<d(g(a),g^2(a)). \tag{ii}$$

But $f(X)\subset g(X)$ by hypothesis; hence, there exists $c\in X$ such that $g(c)=f(a)$. Then (ii) implies $d(g(c),g^2(c))<d(g(a),g^2(a))$, which contradicts (i).

COROLLARY 1. *If g is a continuous mapping of a compact metric space (X,d) into itself such that*

$$d(g^2(x),g^2(y))<d(g(x),g(y)),\qquad g(x)\neq g(y), \tag{3}$$

then g has a unique fixed point.

Proof. Substitute g^2 for f in the statement of the theorem. Since g^2 and g commute and since $g^2(X)\subset g(X)$, g has a fixed point a by our theorem. To see that a is unique, suppose that $g(b)=b$. If $a\neq b$, then $g(a)\neq g(b)$ so that $d(a,b)=d(g^2(a),g^2(b))<d(g(a),g(b))=d(a,b)$; i.e., $d(a,b)<d(a,b)$.

Clearly condition (1) implies condition (3). We thus have Edelstein's theorem as a consequence of Corollary 1. To see that Corollary 1 is in fact a generalization of Edelstein's theorem, consider the following example:

EXAMPLE. Let $X=[0,1]$ and let d be the absolute value metric. Define

$$g\colon X\to X \qquad\text{by}\qquad g(x)=\begin{cases}3x+\frac14 & (0\leqq x\leqq\frac14),\\[2ex] 1 & (\frac14<x\leqq 1).\end{cases}$$

Then $g(x)\geqq\frac14$ for all x so that $g^2(x)=1$, $x\in X$. Thus $|g(x)-g(y)|>0=|g^2(x)-g^2(y)|$ if $g(x)\neq g(y)$, so that condition (3) holds. On the other hand, $d(g(x),g(y))\nless d(x,y)$ when $x,y\in[0,\frac14]$ and $d(g(x),g(y))\ngtr d(x,y)$ if $x,y\in[\frac14,1]$.

We now return to the statement of our theorem. Precisely one of the promised fixed points of g is also a fixed point of f. To show this we apply Edelstein's theorem to the set $F_g=\{x\in X\colon g(x)=x\}$. Note that if f and g commute and $a=g(a)$, then $f(a)=f(g(a))=g(f(a))$; thus $f(F_g)\subset F_g$. Observe also that on the set F_g condition (2) reduces to $d(f(x),f(y))<d(x,y)$.

COROLLARY 2. *Let f and g be commuting mappings of a compact metric space (X, d) into itself such that $f(X) \subset g(X)$ and g is continuous. If condition (2) is satisfied, then there is a unique point a in X such that $a = f(a) = g(a)$.*

Proof. By our theorem, $F_g \neq \varnothing$. Moreover, F_g is a closed subset of X since g is continuous, and therefore F_g is compact. Since $f_{|F_g}$ is a mapping of the compact metric space (F_g, d) into itself satisfying (1), Edelstein's theorem yields a unique point a in F_g such that $a = f(a)$; i.e., there is a unique point a in X such that $a = f(a) = g(a)$.

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Expressing One as a Sum of Odd Reciprocals

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It would seem that the search for a set of distinct positive odd integers whose reciprocals add up to one is a long process. Clearly, the number of reciprocals is odd (otherwise the addition of the reciprocals yields a fraction with an even numerator and an odd denominator) and five are too few (since $1/3 + \cdots + 1/11 < 1$). It is straightforward to verify that seven also are too few, so at least nine numbers are needed. Tedious calculation shows that if nine is possible, five of them must be $\{3, 5, 7, 9, 11\}$. It is not known if a representation involving only nine reciprocals is possible or if, perhaps, the minimum number of required reciprocals is 11 or 13.

In this note we offer a quick determination of 13 odd numbers whose reciprocals add up to 1. We use the identity

$$\frac{1}{3k+2} + \frac{1}{6k+3} + \frac{1}{(3k+2)(6k+3)} = \frac{1}{2k+1}$$

for $k = 1$ and 3, together with the equations $1/5 = 1/7 + 1/21 + 1/105$ and $1/9 = 1/7 - 2/63$ to express 1 as, in turn, the sum of the reciprocals of each of the sets $\{3, 3, 3\}$, $\{3, 5, 9, 45, 5, 9, 45\}$, $\{3, 5, 9, 45, 7, 21, 105, 11, 63, 231, 55, 315, 1155\} = \{3, 5, 7, 9, 11, 21, 45, 55, 63, 105, 231, 315, 1155\}$. ("Chain reaction" substitutions such as this are discussed in considerable detail in [**1**], although without the requirement that all integers be odd.)

Investigation of sets with 11 or 9 elements should be amenable to computer search. Another interesting problem is to determine whether 1 can be expressed as the sum of reciprocals of distinct numbers (odd or even), each of which is the product of exactly two primes.

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Triangulations and Pick's Theorem

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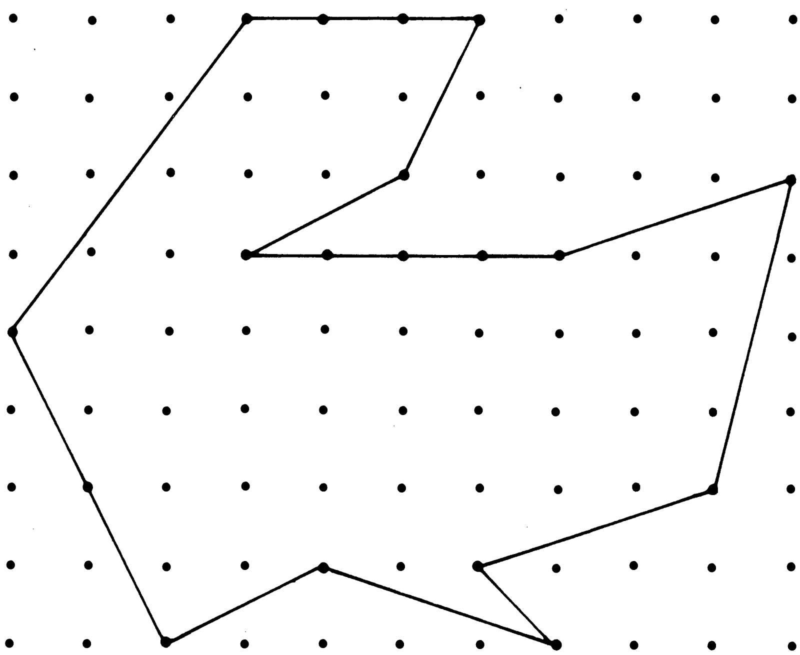
The points of a plane whose coordinates are both integers are called lattice points. The lattice points give rise to surprisingly many interesting problems and results. In 1899, George Pick discovered a remarkable formula for the area of a simple polygon P whose vertices are lattice points ("simple" means that the polygon does not intersect itself):

$$\text{Area} = V_i + \tfrac{1}{2} V_b - 1,$$

where V_i and V_b, respectively, denote the number of lattice points in the interior and on the boundary of P. Observe that V_b includes, in addition to the vertices, any lattice points which occur on the boundary between the vertices.

An interesting proof of Pick's theorem is contained in [3]. The proof centers around showing that the area of a so-called "primitive" triangle is 1/2; a primitive triangle has no lattice points inside or on the boundary except for the (non-collinear) vertices themselves. It is not difficult to convince oneself that any simple polygon P can be decomposed into primitive triangles by appropriately joining up its lattice points with non-intersecting segments. For such a triangulation, Pick's theorem merely gives



PICK'S THEOREM states that the area of any simple polygon whose vertices are lattice points in the plane is given by the sum of the lattice points it encloses (V_i) and one-half the number of lattice points on its boundary (V_b), less one: $A = V_i + V_b/2 - 1$. In this figure, the area is therefore $38 + (18)/2 - 1 = 46$.

the area as one-half the number of primitive triangles. A more general statement — set as a problem in the 1969 William Lowell Putnam Mathematical Competition — would be that every primitive triangulation of P contains the same number of primitive triangles, $2V_i + V_b - 2$. (There is a theorem of N.J. Lennes which says that, for any given polygon, there is a triangulation such that $V_i = 0$. For this theorem and the consequent formula $T = V_b - 2$, see **[5]**. For a number of other interesting applications, see **[6]**.)

We see, then, that Pick's theorem is not really about area, but is a combinatorial result which essentially belongs to topology. It is not important that the lattice points of P occur at regular intervals across the figure and that all primitive triangles have the same area. The same *number* of (primitive) triangles would occur if V_i and V_b points were chosen arbitrarily, respectively, in the interior and on the boundary of a polygon P (the vertices among the V_b points). In fact, the triangulation need not even begin with a polygon or be carried out with straight segments. We require only that each region in the completed decomposition of a simple closed curve P be bounded by three arcs and that no arcs cross. It is our purpose in the present article to show that, as one might expect, Pick's topological result follows from an application of the most famous of all topological formulas, Euler's $V - E + F = 2$. From the *number* of primitive triangles, in a rectangle, it is a simple matter to deduce (we will do it below) that the area of a primitive triangle in a lattice is 1/2. With each of the $2V_i + V_b - 2$ primitive triangles having area 1/2, Pick's formula follows immediately. (A recent similar derivation of Pick's formula from Euler's formula is given in **[7]**; and the difficulties of generalizing Pick's theorem to three dimensions are discussed briefly in **[8]**.)

Let us prove, then, that the number of triangles in a primitive triangulation of a polygon P is $T = 2V_i + V_b - 2$. Imagine that an identical copy is superimposed exactly on top of the triangulation and that the two configurations are glued together along the matched edges of p. Let the copies be elastic and suppose that the enclosure is inflated to a sphere. Euler's formula now yields the desired result: each copy of the triangulation contributes T triangles, giving a total of $F = 2T$ faces on the sphere. Since the interior vertices of the triangulation occur in each copy while the boundary vertices are glued together in pairs, the total number of vertices on the sphere is $V = 2V_i + V_b$. Counting the $2T$ triangles at 3 edges apiece, we obtain a total of $6T$ edges. However, since each edge bounds two triangles, the actual number of edges is only $E = 3T$. Now Euler's formula gives

$$V - E + F = 2V_i + V_b - 3T + 2T = 2.$$

Thus

$$T = 2V_i + V_b - 2.$$

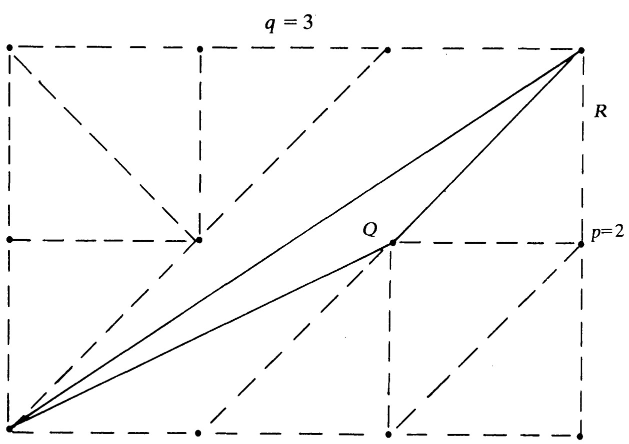


FIGURE 1

(In *Mathematics, The Man-Made Universe*, ([4]), Sherman Stein gives an exercise to show that if the polygon P is a triangle, then T depends solely on V_i. The clever notion of glueing together two elastic copies of the configuration is also given by Stein, but it is not used explicitly to deduce the general formula for T.)

It remains to show that the area of each primitive triangle is 1/2. Since the area of a primitive triangle is not altered by moving the origin to one of its vertices, we may, without loss of generality, consider the vertices of a primitive triangle to be (0, 0), (r, s), and (m, n). Its area is given by

$$\Delta = |rn - sm|/2.$$

Since r, s, m, n, are integers and the area is non-zero, its magnitude cannot be less than 1/2.

Consider now any lattice rectangle R (one whose sides lie on the lattice lines) which encloses a primitive triangle Q. Suppose it has dimensions p and q and that the part(s) of R around the primitive triangle Q is triangulated to provide, with Q, a complete primitive triangulation Z of R (see FIGURE 1). For the rectangle R, we have $V_i = (p-1)(q-1)$ and $V_b = 2p + 2q$. Our result above, then, yields the number of primitive triangles in Z to be

$$T = 2V_i + V_b - 2 = 2(p-1)(q-1) + 2p + 2q - 2 = 2pq.$$

Thus we have $2pq$ primitive triangles, each of area not less than 1/2. But the area of R is pq. Consequently each of the primitive triangles (including Q) must have area 1/2, lest their total area exceed pq.

Acknowledgements

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The Equation $\phi(x) = k$

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Most elementary textbooks give only two examples in their problem sections which deal with solutions of the equation $\phi(x) = k$, where ϕ is the usual Euler function, namely, $\phi(x) = 14$, which has no solution, and $\phi(x) = 24$, which has ten solutions. This led my class on elementary number theory to a discussion concerning what properties of numbers lead to no solutions and what properties lead to a relatively large number of solutions. The starting points were the basic facts that $\phi(x) = k$ has no solution if k is odd and greater than one, and that those cases in which $\phi(x) = k$ had a large number of solutions occurred when k was divisible by a large power of two. Next, the solutions were divided into two types, namely: (1) x is a prime power p^r and (2) $x = ab$ where $(a, b) = 1$ and $a > 1$, $b > 1$. Can the equation have two solutions of the first type? This would require $\phi(p^r) = \phi(q^s)$ where p and q are distinct primes with $p < q$, (say). Hence $p^{r-1}(p-1) = q^{s-1}(q-1)$. If $s > 1$ then $q \mid p-1$, a contradiction. Hence $s = 1$ and $q = 1 + p^{r-1}(p-1)$. One concludes that there are at most two solutions since the larger prime power if it exists, is uniquely determined by the smaller one. That two solutions can exist is illustrated by the case $p = 3$, $r = 2$, $q = 7$ with $\phi(7) = \phi(9) = 6$. Note that in particular q is a Fermat prime when $p = 2$.

Next, we examine solutions of the second type. Let $x = ab$, $(a, b) = 1$, $1 < a < b$. Then $\phi(a)\phi(b) = k$. Hence, either $\phi(a)$ and $\phi(b)$ are both even or $\phi(a) = 1$, i.e., $a = 2$. In the latter case, b is odd, yielding the two solutions $\phi(b) = \phi(2b) = k$. In the former case $\phi(a)$ and $\phi(b)$ are both even, so that k is a product of two even integers. This leads to the following procedure for obtaining further solutions. Write $k = 2u \cdot 2v$ and solve $\phi(a) = 2u$, $\phi(b) = 2v$. We obtain solutions of $\phi(x) = k$ by multiplying together relatively prime solutions from each of the two equations. This gives an algorithm for finding all solutions of $\phi(x) = k$, when we know all solutions of $\phi(x) = l$, with $l < k$. For example consider $\phi(x) = 24 = 2^3 \cdot 3$. There are no solutions of the form $x = p^r$; ($2^3 \cdot 3 = p^{r-1}(p-1)$ cannot be solved). Hence we write 24 as the product of two even integers in all possible ways; in this case $24 = 2 \cdot 12 = 4 \cdot 6$. Now $\phi(a) = 2$ has solutions $a = 3, 4, 6$; $\phi(b) = 12$ has solutions $b = 13, 21, 26, 28, 36, 42$; these yield solutions $x = 35, 45, 56, 70, 72, 84, 90$. In the same way $24 = 4 \cdot 6$ yields new solutions $x = 39, 52, 78$.

Summarizing, $\phi(x) = k$ can only have a large number of solutions when k is divisible by a large power of 2. In fact, our discussion has shown that if $k = 2n$, where n is odd then $\phi(x) = k$ has at most 4 solutions, the four solutions occurring in the case where there is an odd prime p and an integer r such that $1 + p^{r-1}(p-1)$ is equal to a prime q. In this case $\phi(p^r) = \phi(q) = \phi(2p^r) = \phi(2q)$. In the previously cited example $p = 3$, $r = 2$, $q = 7$, we have $\phi(9) = \phi(7) = \phi(18) = \phi(14) = 6$.

Amongst the questions to which the above discussion leads is the following: if k is divisible by a large power of 2, does $\phi(x) = k$ of necessity have a large number of solutions? A natural starting point is $k = 2^r p$ where p is an odd prime. It is easy to give examples of primes p such that for $k = p, 2p, 4p$ the equation $\phi(x) = k$ has no solution. Hence, for a given prime p we look for a smallest m for which $\phi(x) = 2^m p$ has a solution. The minimality of m implies that the only solutions are of the first type, i.e., $x = q^r$ or $x = 2q^r$, where q is a prime. This implies $q^{r-1}(q-1) = 2^m p$. There are only two possibilities: $r = 1$ and $2^m p + 1$ is prime or $r = 2$, p is a Fermat prime F_k and $m = 2^k$. If we exclude Fermat primes we find that the least m for which $\phi(x) = 2^m p$ has a solution is the least m for which $1 + 2^m p$ is a prime. In turn this leads to the question as to when, if ever, the first prime appears in the sequence $1 + 2p, 1 + 4p, 1 + 8p, \ldots$.

Primes of the form $1 + 2^k p$ (more generally, $1 + 2^k n$ with n odd) are the type of primes which are possible divisors of the Fermat numbers $F_m = 1 + 2^{2^m}$. In [1], R. M. Robinson lists primes of the form $1 + 2^k n$ where $n = 1, 3, 5, \ldots, 99$ and $k \leqq 512$. A striking case appears in this table at $n = 47$. Not a single prime was listed. The example was given to Professor H. C. Williams of the computer science department at the University of Manitoba to make a computer search — the result was that $1 + 2^k(47)$ is not prime for $k \leqq 582$ but that $1 + 2^{583}(47)$ is prime. It now seemed plausible that there are primes p which are not Fermat primes but for which $1 + 2^n p$ is composite for all n. If this conjecture were true then for such a prime the equation $\phi(x) = 2^n p$ would have no solution for every positive integeral n. In what follows, we prove this conjecture, not just for some primes, but for infinitely many primes.

THEOREM. *There exist infinitely many primes p such that the equation $\phi(x) = 2^n p$ has no solutions for every positive integer n.*

Proof. We will show that there exist infinitely many primes p which are not Fermat primes such that $2^r p + 1$ is composite for every positive integer r. More specifically, every term of the sequence is divisible by at least one of the primes $3, 5, 17, 257, 641, 65537, 6700417$. To do this, consider the set of congruences

$$2y+1\equiv 0 \bmod 5,\qquad 2^2y+1\equiv 0 \bmod 3,\qquad 2^3y+1\equiv 0 \bmod 17,$$
$$2^7y+1\equiv 0 \bmod 257,\qquad 2^{15}y+1\equiv 0 \bmod 65537,\quad 2^{31}y+1\equiv 0 \bmod 641,$$
$$2^{63}y+1\equiv 0 \bmod 6700417.$$

By the Chinese Remainder Theorem, the general solution is $y = a + bt$ where $a = (2^{32}-1)(137)(33343)(641)+2$ and $b = 2^{64}-1$. Also $(a, b) = 1$. By Dirichlet's theorem, there are infinitely many integeral values of t for which the corresponding y is prime. Also the number of primes less than or equal to x in the arithmetic progression $a + bt$ is asymptotic to $(1/\phi(b))\,(x/\log x)$. However, the number of Fermat numbers $F_n = 1+2^{2^n}$ which are less than x is trivially $o(x/\log x)$. Hence, infinitely many primes in the sequence are not Fermat primes.

To complete the proof we will use the following technical lemma: *If p is a prime and m is the least integer such that $p\,|\,2^mn+1$, then $p\,|\,2^xn+1$ iff $x\equiv m \bmod u$, where u is the order of* 2 (mod p). *This follows directly from the nearly obvious fact that if p is a prime such that $p\,|\,n+1$, then $p\,|\,2^rn+1$ iff $2^r\equiv 1 \bmod p$.*

Now let p be a non-Fermat prime value of y satisfying the congruences. First note that for the primes $3, 5, 17, 257, 641, 65537, 6700417$, the order of 2 is $2, 4, 8, 16, 64, 32, 64$ respectively. Hence, by the corollary to Lemma 1: $x\equiv 1 \bmod 4$ implies $5\,|\,2^xp+1$; $x\equiv 0 \bmod 2$ implies $3\,|\,2^xp+1$; $x\equiv 3 \bmod 8$ implies $17\,|\,2^xp+1$; $x\equiv 7 \bmod 16$ implies $257\,|\,2^xp+1$; $x\equiv 15 \bmod 32$ implies $65537\,|\,2^xp+1$; $x\equiv 31 \bmod 64$ implies $641\,|\,2^xp+1$, and $x\equiv 63 \bmod 64$ implies $6700417\,|\,2^xp+1$. The set of congruences for x encompasses all the congruence classes mod 64. Hence every integer x satisfies one of these congruences. This completes the proof.

If we are to obtain small primes for which the theorem holds we require a set of primes for which the order of 2 is small and an appropriate covering of the integers by arithmetic progressions. This can be accomplished by finding an integer n for which both n and 2^n-1 have many divisors. For instance, $n = 24$ and $2^{24}-1 = 3^3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241$. Following the same argument as in the proof of the theorem but using the primes $3, 5, 7, 13, 17, 241$, a computer search of all possibilities has yielded 48 arithmetic progressions of the form $a + 5592405\,t$, and the smallest prime in any of these is 271,129. The next promising value of n is $n = 36$ with $2^{36}-1 = 3^2\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73\cdot 109$. A similar computation in this case produced 576 arithmetic progressions but did not yield a smaller prime. However, it did yield the number $78557 = 17\cdot 4621$. In this case $\phi(x) = 2^n(78557)$ has no solution for any n. The computer program was carried out by B. Wolk.

In general, the treatment of the equation $\phi(x) = 2^kn$ for n odd is essentially the same as the case for which $n = p$, a prime. We have already seen that we must avoid Fermat primes for our theorem to be true when $n = p$. If n is composite and odd, the necessary and sufficient conditions for $\phi(x) = 2^kn$ to have no solutions for every k is that 2^kn+1 is composite for every k, and that $2^kn \neq Q^r(Q-1)$ for any prime Q and any positive integer k. This latter condition is easily verified for any specific n. We now examine some special cases.

(1) $n = p^t$, p prime. Here $2^kp^t = Q^r(Q-1)$ implies $Q = p$, $r = t$, $Q-1 = 2^k$. Again, this implies that p is a Fermat prime. Hence, again we must avoid Fermat primes.

(2) $n = pq$, p and q odd primes, $p < q$. In this case $2^kpq = Q^r(Q-1)$ implies $Q = q$, $r = 1$; $Q-1 = 2^kp$. Hence, $q = 2^kp+1$ is the case we must avoid.

The prime $p = 271,129$ and the number $78,557$ are the smallest numbers that are known which satisfy our theorem. The near miss $p = 47$ encourages a computer search for the smallest integer for which our theorem holds. The task appears quite formidable.

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The Most "Elementary" Theorem of Euclidean Geometry

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The late M. L. Urquhart (1902–1966) formulated what he believed to be the most "elementary" theorem of Euclidean geometry. David Elliott [1, p. 132] describes the event: "I recall him coming into my office one day and asking me what I thought was the most elementary theorem of Euclidean geometry. Since it was obvious that whatever I said was going to be incorrect, he rapidly gave the following theorem:

'*Let AC and AE be two straight lines which intersect at A. Let B be a point on AC, D a point on AE, and suppose that BE and CD intersect at F. If $AB + BF = AD + DF$, then $AC + CF = AE + EF$.*'

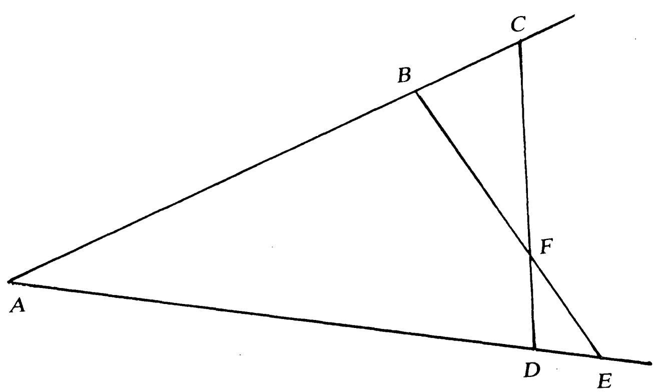
Urquhart considered this to be 'the most elementary theorem' since it involves only the concept of straight line and distance. The proof of this theorem by purely geometrical methods is not elementary. Urquhart discovered this result when considering some of the fundamental concepts of the theory of special relativity."

Curiously enough, Urquhart's description of his theorem has been transposed by Howard Grossman [2] into "the most interesting theorem of Euclidean geometry." Grossman proves the theorem, but in the same volume of the Journal of the Australian Mathematical Society referred to above George Szekeres [3, p. 159] gives a proof by Basil Rennie which is based on the following lemma:

Given a circle and two points A and C outside the circle, the line AC will be a tangent if the distance AC equals the difference between (or the sum of) the length of a tangent from A and the length of a tangent from C.

In his paper, which deals with Minkowski space-time, Szekeres gives the theorem which led Urquhart to his 'most elementary' theorem.

I attempted to find a proof of the Urquhart theorem which did not bring in the circle which, given the hypotheses of the theorem, can be shown to touch the sides, all produced, of the quadrilateral $ACFE$, and noted that an equivalent to Urquhart's theorem is the following:



URQUHART'S "ELEMENTARY" THEOREM: If $AB + BF = AD + DF$, then $AC + CF = AE + EF$. While this theorem deals only with straight lines and distances, the proof seems inevitably to involve circles, such as the one which can be shown to touch the sides, all produced, of the quadrilateral *ACFE*.

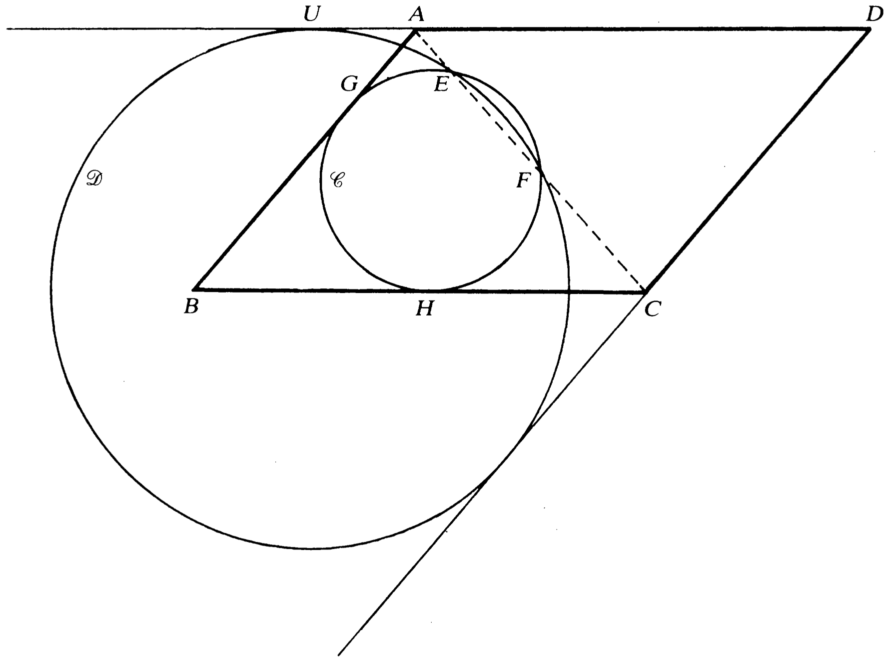


FIGURE 1.

If C and E are points on an ellipse with foci A and F, then $B = AC \cap EF$ *and* $D = AE \cap CF$ *lie on a confocal ellipse.*

Proving this seemed inevitably to bring in a circle, so I applied the method of reciprocal polars, invoking the contentious shades of Poncelet, Gergonne, Plücker and even Möbius, using a circle center A with respect to which polars are taken. I obtained the following equivalent theorem, unknown to me:

ABCD is a parallelogram, and a circle $\mathscr{C}$ *touches AB and BC and intersects AC in the points E and F. Then there exists a circle* $\mathscr{D}$ *which passes through E and F and touches AD and DC.*

Without venturing to call this 'the most elementary theorem of circle geometry' it is clear that this is not a trivial theorem. To prove it, let $\mathscr{D}$ be a circle through E and F which touches DA at U (FIGURE 1), and suppose that $\mathscr{C}$ touches AB at G and BC at H. We wish to show that $\mathscr{D}$ touches CD.

Since $\mathscr{D}$ passes through E and F, the point A is on the radical axis of $\mathscr{C}$ and $\mathscr{D}$, and $AG = AU$. We note that C is also on the radical axis of these two circles, and so the length of the tangent from C to $\mathscr{D}$ is equal to CH. We now have:

$$\begin{aligned} DU &= DA + AG = DA + AB - GB \\ &= DA + AB - BH \\ &= DA + AB - [BC - CH] \\ &= AB + CH, \end{aligned}$$

so that

$$DC = AB = DU - CH,$$

and, using Rennie's lemma, this proves that $\mathscr{D}$ touches CD.

The investigation therefore seemed to be a circular one, but it did produce an attractive circle theorem I had not seen before.

It should be mentioned in conclusion that Yaglom [**4**, p. 92] gives a criterion for four oriented lines, not all parallel, to touch an oriented circle or to pass through one point, and this criterion embraces the Urquhart theorem.

References

[1] D. Elliott, M. L. Urquhart, J. Austral. Math. Soc., 8 (1968) 129–133.

[2] H. Grossman, Urquhart's Quadrilateral Theorem, The Mathematics Teacher, 66 (1973) 643–644.

[3] G. Szekeres, Kinematic geometry, M. L. Urquhart in memoriam, J. Austral. Math. Soc., 8 (1968) 134–160.

[4] I. M. Yaglom, Complex Numbers in Geometry, Academic Press, New York, 1968.

An Elementary Example of a Transcendental p-adic Number

GLEN H. SUTER

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This note gives a simple example of a transcendental p-adic number. In particular we show that $\Sigma_{i=1}^{\infty} p^{i!}$ is transcendental. The proof of this fact is merely the p-adic analogue of the proof given by Liouville for the fact that $\Sigma_{i=1}^{\infty}(-1)^i 2^{-i!}$ is transcendental in the real numbers. All methods used are elementary.

THEOREM. *$\Sigma_{i=1}^{\infty} p^{i!}$ is a transcendental p-adic number.*

Proof. Let $\alpha = \Sigma_{i=1}^{\infty} p^{i!}$ and let $\alpha_k = \Sigma_{i=1}^{k} p^{i!}$. Since α's p-adic representation is not repeating we know that α is not a rational number. Assume α is algebraic. Then there exists a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients such that $f(\alpha) = 0$, $f(x)$ is irreducible over the rationals and the degree of $f(x)$ is greater than one.

Clearly

$$f(\alpha) - f(\alpha_k) = a_n(\alpha^n - (\alpha_k)^n) + a_{n-1}(\alpha^{n-1} - (\alpha_k)^{n-1}) + \cdots + a_1(\alpha - (\alpha_k)).$$

And since $\alpha - \alpha_k$ divides $\alpha^j - (\alpha_k)^j$ for $1 \leqq j \leqq n$, we can conclude that $\alpha - \alpha_k$ will divide $f(\alpha) - f(\alpha_k)$. Moreover the coefficients of $f(x)$ are integers. Hence it follows that $\|f(\alpha) - f(\alpha_k)\|_p \leqq \|\alpha - \alpha_k\|_p$.

Let Δ be a positive integer such that p^{Δ} is greater than $\Sigma_{i=0}^{n} |a_i|$. The highest power of p appearing in $f(\alpha_k)$ is $(p^{k!})^n = p^{n(k!)}$. Say $f(\alpha_k) = a_n p^{n(k!)} + b_1 p^w + \cdots + a_0$. Here b_1 is a sum of the a_i's and p^w is the next highest power in the expanded $f(\alpha_k)$. Hence

$$f(\alpha_k) \leqq |f(\alpha_k)| \leqq p^{\Delta} p^{n(k!)} + p^{\Delta} p^w + \cdots + p^{\Delta} \leqq \sum_{i=0}^{n(k!)+\Delta} p^i.$$

Therefore $f(\alpha_k) \leqq (1 - p^{n(k!)+\Delta+1})/(1-p) \leqq p^{n(k!)+\Delta+1}$. Hence

$$p^{-(n(k!)+\Delta+1)} \leqq \|f(\alpha) - f(\alpha_k)\|_p \leqq \|\alpha - \alpha_k\|_p \leqq p^{-(k+1)!}.$$

Therefore for each k, $1 \leqq p^{[n-(k+1)]k!} \cdot p^{\Delta} \cdot p$. But $\mathrm{Lim}_{k\to\infty} p^{[n-(k+1)]k!} p^{\Delta} \cdot p = 0$. This contradiction establishes the result.

This paper was written while the author was partially supported on NSF Grant GP-44372.

PROBLEMS

DAN EUSTICE, Editor
LEROY F. MEYERS, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before August 1, 1976.

963. Characterize convex quadrilaterals with sides a, b, c, and d such that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0$$

[*Hüseyin Demir, Ankara, Turkey.*]

964. Show that every positive integer k, $k < n!$, is a sum of fewer than n distinct divisors of $n!$. [*Paul Erdös, Hungarian Academy of Science.*]

965. Find all polynomials $P(x)$ such that $P(F(x)) = F(P(x))$, $P(0) = 0$, where $F(x)$ is a given function satisfying $F(x) > x$ for all $x \geqq 0$. [*Bernard B. Beard,* 1975 *U.S.A. Mathematical Olympiad Team.*]

966. A point P lies in the interior of a rectangle of sides a and b.
(i) Find a, b, and P so that all eight distances from P to the four vertices and the four sides are positive integers.
(ii) Find an example of a square where seven of the distances are integers.
(iii)* Can all eight distances be integers for a square?
[*Clayton W. Dodge, University of Maine at Orono.*]

967. Let ABC be a triangle inscribed in a circle with the internal bisectors of the angles B and C meeting the circle again in the points B_1 and C_1 respectively. (i) If $B = C$, prove $BB_1 = CC_1$. (ii) Characterize triangles ABC for which $BB_1 = CC_1$. Do these results hold if BB_1 and CC_1 are the external bisectors? [*K. R. S. Sastry, Addis Ababa, Ethiopia.*]

ASSISTANT EDITORS: DON BONAR, *Denison University*; WILLIAM A. MCWORTER, JR., *The Ohio State University.*
We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgement of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

968. A point in the plane is called rational if both of its coordinates are rational numbers. Show that $x^2+y^2=2$ has an infinite number of rational solutions. [*Sidney Penner and H. Ian Whitlock, Bronx Community College.*]

969. A cube can be unfolded into a polyomino of order six in the form of a Latin cross.
(i) Show that five congruent Latin crosses can cover the surface of the cube without overlap.
(ii)* Can the surface of the cube be covered with seven congruent polygons? [*Veit Elser, San Jose, California.*]

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q631. Solve the differential equation $(xD^4 - axD + 3a)y = 0$. [*M. S. Klamkin, University of Waterloo.*]

Q632. A tetrahedron and an octahedron are built from a common stock of equilateral triangles. The tetrahedron holds a quart; what does the octahedron hold? [*F. David Hammer, Stockton State College.*]

Solutions

The Series Converges **January 1975**

922. Let $\{x_n\}$ be a sequence of nonnegative numbers satisfying

$$\sum_{n=0}^{\infty} x_n x_{n+k} \leqq Cx_k$$

for some constant C and $k = 0, 1, 2, \ldots$. Prove that Σx_n converges. (Is the result still true if $k = 0, 1, 2, \ldots$ is replaced with $k = 1, 2, \ldots$?—Ed.) [*Alan Schwartz, University of Missouri–St. Louis.*]

Solution: If the hypothesis holds for $k = 0, 1, 2, \ldots$ then the partial sums $\Sigma_{n=0}^{q} x_n$ are bounded:

$$\left(\sum_{n=0}^{q} x_n\right)^2 = \left(\sum_{i=0}^{q} x_i\right)\left(\sum_{j=0}^{q} x_j\right) = \sum_{n=0}^{q} x_n^2 + 2\sum_{i<j} x_i x_j$$

$$\leqq 2\left\{\sum_{n=0}^{\infty} x_n^2 + \sum_{n=0}^{\infty} x_n x_{n+1} + \sum_{n=0}^{\infty} x_n x_{n+2} + \cdots + \sum_{n=0}^{\infty} x_n x_{n+q}\right\}$$

$$\leqq 2C\{x_0 + x_1 + x_2 + \cdots + x_q\} = 2C\sum_{n=0}^{q} x_n.$$

Hence the series Σx_n converges.

If we show that Σx_n^2 converges, then our hypothesis again holds for $k = 0, 1, 2, \ldots$, and Σx_n converges as before. Assume there is a $j \geqq 1$ for which $x_j > 0$; if there is no such j, the conclusion is obvious. Since $\Sigma x_n x_{n+k} \leqq C x_k$, taking only the jth term from this series, we see that $x_j x_{j+k} \leqq C x_k$ for $k = 1, 2, \ldots$. Thus $x_{j+k} \leqq C x_k / x_j$ since $x_j > 0$. Multiplying by x_{j+k} and summing on k, we find

$$\sum_{k=1}^{\infty} x_{j+k}^2 \leqq \frac{C}{x_j} \sum_{k=1}^{\infty} x_k x_{k+j} \leqq C^2.$$

It follows that Σx_n^2 converges, completing the proof.

MICHAEL B. GREGORY
University of North Dakota

Also solved by M. T. Bird, R. P. Boas, Alfred Brousseau, Richard Groeneveld, G. A. Heuer, Vaclav Konecny (Czechoslovakia), Graham Lord, Jim McHutchion, John D. O'Neill, Paul Osterhus, U. V. Mallikharjuna Rao (India), J. M. Stark, Phil Tracy, Ken Yocom, and the proposer.

The second part of M. B. Gregory's solution will also show that the hypothesis $k = 0, 1, \ldots$ can be replaced by $k = m, m+1, \ldots$. The second part was also solved by M. T. Bird, R. P. Boas, Richard Groeneveld, Vaclav Konecny, J. M. Stark, and Phil Tracy.

Powers of Roots **January 1975**

923. If r and s are roots of $x^2 + px + q = 0$, where p and q are integers with $q \mid p^2$, then $(r^n + s^n)/q$ is an integer for $n = 2, 3, \ldots$. [*Aron Pinker, Frostburg State College.*]

Solution: Since r and s are roots of the given equation, we have

$$r^n = -pr^{n-1} - qr^{n-2}$$

and

$$s^n = -ps^{n-1} - qs^{n-2}.$$

Hence $r^n + s^n = -p(r^{n-1} + s^{n-1}) - q(r^{n-2} + s^{n-2})$. Now if $n = 2$ we have $r^2 + s^2 = -p(r+s) - 2q = p^2 - 2q$. Hence $r^2 + s^2$ is an integer which is divisible by q, since $p^2 - 2q$ is an integer and $q \mid p^2$; the desired result follows by induction on n.

KENNETH M. WILKE
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Editor's Comment. The relation for $r^n + s^n$ in the solution above shows inductively the generalization that $q^{[n/2]} \mid (r^n + s^n)$. This was also observed in the solutions of L. Carlitz, F. D. Hammer and Graham Lord.

Also solved by Leon Bankoff, G. E. Bergum, Alfred Brousseau, L. Carlitz, Deb Prasad Choudhury (India), Robert H. Cornell, Roy E. DeMeo, Guy M. DePrimo, Clayton W. Dodge, Michael W. Ecker, Thomas E. Elsner, Stanley Fox, Donald C. Fuller, Ralph Garfield, Leumas Giessel, Reinaldo E. Giudici (Venezuela), Michael Goldberg, M. G. Greening (Australia), M. B. Gregory, Richard A. Groeneveld, F. D. Hammer, J. D. Hiscocks (England), M. S. Klamkin, Vaclav Konecny (Czechoslovakia), P. N. Kulkarni (India), Henry S. Lieberman, Peter A. Lindstrom, Graham Lord, Jim McHutchion, Joseph E. Mueller, M. Ram Murty & V. Kumar Murty (Canada), William Nuesslein, John D. O'Neill, Paul Osterhus, Leonard L. Palmer, F. D. Parker (Canada), Albert J. Patsche, Willis B. Porter, Bob Prielipp, U. V. Mallikharjuna Rao (India), James V. Rauff, Sally Ringland, John M. Samoylo, Roland F. Smith, W. Allen Smith, James M. Sobota, Marius Solomon, Eric Sturley, Temple University Problem Solving Group, S. P. Tsatsanis (Canada), Gillian W. Valk, Edward T. Wang (Canada), Alan Wayne, Ken Yocom, and the proposer.

Counting n-Tuples **January 1975**

924. How many n-tuples, $(S_1, \ldots, S_n)$, exist with $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq V$, where V is a set of k elements? [*J. Michael McVoy and Anton Glaser, Pennsylvania State University.*]

Solution: Let F be the set of functions from V into $\{1, 2, \ldots, n+1\}$, and let G be the set of n-tuples $(S_1, S_2, \ldots, S_n)$ with $S_1 \subseteq S_2 \subseteq \cdots S_n \subseteq V$.

There is a natural one to one correspondence between F and G. For each $f \in F$, we can construct a unique n-tuple in G by setting $S_j = \{x: f(x) \leq j\}$, and, conversely, for each n-tuple in G, we can construct a unique $f \in F$ by setting $f(x) = n+1$ if $x \notin S_n$ and $f(x) = \min\{j: x \in S_j\}$ if $x \in S_n$.

Therefore, the number of n-tuples in G equals the number of functions in F. Since there are $(n+1)^k$ functions in F, there are $(n+1)^k$ n-tuples in G.

WILLIAM NUESSLEIN
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Also solved by Brother Alfred Brousseau, Michael Capobianco, Deb Prasad Choudhury (India), M. G. Greening (Australia), Myron Hlynka, N. J. Kuenzi, Graham Lord, M. Ram Murty & V. Kumar Murty (Canada), John D. O'Neill, Paul Osterhus, Ken Rebman, David Singmaster (London), Robert G. Smith, Temple University Problem Solving Group, Ken Yocom, and the proposers.

Inscribed Octagons **January 1975**

925. a. Prove that any nonself-intersecting cyclic octagon is such that the sum of any four nonadjacent interior angles is 3π. [*Julius G. Baron, Rye, New York.*]

b. An octagon is inscribed in a circle with vertices on any four diameters. Show that each alternate pair of exterior angles is complementary. [*Thomas E. Elsner, General Motors Institute.*]

Solution: To obtain mild generalizations, let $\{P_i: i = 1, 2, \ldots, r\}$ be the clockwise sequence of vertices of an inscribed nonself-intersecting r-gon. Let m_i be the measure of the interior angle whose vertex is P_i, and let a_i be the measure of the arc in which that angle is inscribed.

a. Suppose that $r = 2n$. Then for $k = 1, 2, \ldots, n$ we have $m_{2k} = (2\pi - a_{2k})/2$. Then $\Sigma_{k=1}^{n} m_{2k} = (n-1)\pi$. Therefore any nonself-intersecting cyclic $2n$-gon is such that the sum of any n consecutive nonadjacent interior angles is $(n-1)\pi$.

b. Suppose that $r = 4s$. From the previous results, with $n = 2s$, the sum of any $2s$ consecutive nonadjacent interior angles is $(s-1)\pi$. Let P_k and P_{2s+k} be diametrically opposite points ($k = 1, 2, \ldots, s$), so that $m_{2k} = m_{2s+2k}$. Then $\Sigma_{k=1}^{s}(\pi - m_{2k}) = \pi s - (2s-1)\pi/2 = \pi/2$. Therefore in any nonself-intersecting cyclic $4s$-gon with vertices on any $2s$ diameters, the sum of any s consecutive nonadjacent exterior angles is $\pi/2$.

ALAN WAYNE
Pasco-Hernando Community College

Also solved by G. E. Bergum, Brother Alfred Brousseau, Deb Prasad Choudhury (India), Clayton W. Dodge, Frank M. Eccles, Donald C. Fuller, J. Garfunkel, Michael Goldberg, M. G. Greening (Australia), M. S. Klamkin (Canada), J. D. Hiscocks (England), Vaclav Konecny (Czechoslovakia), Lew Kowarski, Graham Lord, John M. O'Malley, Jr., Sally Ringland, K. R. S. Sastry (Ethiopia), Brian Smithgall, Charles W. Trigg, William Wernick, and the proposers.

The Longest Swim **January 1975**

926. A swimmer can swim with speed v in still water. He is required to swim for a given length of time T in a stream whose speed is $r < v$. If he is also required to start and finish at the same point, what is the longest path (total arc length) that he can complete? Assume the path is continuous with piecewise continuous first derivatives. [*Melvin F. Gardner, University of Toronto.*]

Solution: If $\theta(t)$ denotes the angle heading of the swimmer with respect to the stream velocity, then

$$\dot{x} = \frac{dx}{dt} = v\cos\theta + r, \qquad \dot{y} = \frac{dy}{dt} = v\sin\theta.$$

The length L of a closed path swum in time T is then given by

$$L = \int_0^T \{\dot{x}^2 + \dot{y}^2\}^{1/2}\,dt = \int_0^T \{v^2 + 2vr\cos\theta + r^2\}^{1/2}\,dt$$

$$= \int_0^T \{v^2 - r^2 + 2r\dot{x}\}^{1/2}\,dt.$$

Applying the Schwarz-Buniakowski inequality and noting that $\int_0^T \dot{x}dt = 0$,

$$L^2 \leqq \int_0^T \{v^2 - r^2 + 2r\dot{x}\}dt \cdot \int_0^T dt = T^2(v^2 - r^2),$$

with equality iff $\dot{x}$ = constant. Thus $L_{\max} = T(v^2 - r^2)^{1/2}$ for a back and forth segment path perpendicular to the stream velocity.

We can also find the closed path of minimum length for a given time T. Since

$$v^2 + 2vr\cos\theta + r^2 \geqq (v + r\cos\theta)^2,$$

$$L \geqq \int_0^T (v + r\cos\theta)dt = \int_0^T \{v + r(\dot{x} - r)/v\}dt = (v^2 - r^2)T/v$$

with equality iff $\cos^2\theta = 1$. Thus, $L_{\min} = (v^2 - r^2)T/v$ for a back and forth segment path parallel to the stream velocity.

The above results are generalized for the flight of an aeroplane in a three-dimensional irrotational wind field in a paper *On extreme length flight paths* submitted for publication.

M. S. Klamkin
University of Waterloo

Also solved by Thomas E. Elsner, Michael Goldberg, Vaclav Konecny, Lew Kowarski, William Nuesslein, David Wright, Harald Ziehms, and the proposer.

Pick's Formula **January 1975**

927. Pick's formula for the area of polygonal regions whose vertices are lattice points is $\frac{1}{2}b + i - 1$ where b is the number of lattice points on the boundary and i is the number of lattice points in the interior. Show that no such formula exists for the volume of polyhedra whose vertices are lattice points even if we allow as variables, in addition to b and i, e = the number of edges, f = the number of faces, and i' = the number of lattice points in the interior of the faces. [*Roy Dubisch, University of Washington.*]

Solution: In a paper *On the volume of lattice polyhedra* (Proc. Lond. Math. Soc.), J. E. Reeve shows, with the tetrahedron whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,r)$ where r is a positive integer, that no such formula can exist. The tetrahedron contains no other lattice points yet its volume can increase without changing the variables b, e, f, or i'.

See also Martin Gardner's *Mathematical games*, Scientific American, May, 1965, pages 123–126 and I. Niven & H. S. Zuckerman, *Lattice points and polygonal area*, American Math. Monthly, 74 (1967), 1195–1200.

Robert H. Cornell
Exeter, New Hampshire

Also solved by Alfred Brousseau, F. D. Hammer, Jim McHutchion, and the proposer.

928. If k is a positive integer, prove that

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{j=1}^{n} \cot^k\left(\frac{1}{j}\right) = \frac{1}{k+1}.$$

[*Norman Schaumberger, Bronx Community College.*]

Solution: Euler's Summation Formula (see Apostol's *Mathematical Analysis,* p. 201) is

$$\sum_{j=a+1}^{n} f(j) = \int_a^n f(x)dx + \int_a^n (x-[x])f'(x)dx,$$

where a is an integer and $f'(x)$ is continuous for $a \leqq x \leqq n$. Applying this, the given limit becomes

$$\lim_{n\to\infty} \left\{ \frac{\int_1^n \cot^k\left(\frac{1}{x}\right) dx}{n^{k+1}} + \frac{\int_1^n (x-[x]) \frac{d}{dx}\left(\cot^k\left(\frac{1}{x}\right)\right) dx}{n^{k+1}} \right\}.$$

Since $\int_1^n \cot^k (1/x)dx \to \infty$ as $n \to \infty$, we may apply L'Hospital's rule to the first term getting a limit of $1/(k+1)$. Since $0 \leqq x-[x] < 1$, the second term is bounded by $\cot^k (1/n)/n^{k+1}$ which approaches zero as $n \to \infty$.

KEN YOCOM
South Dakota State University

Also solved by Mangho Ahuja, Leon Gerber, Richard A. Groeneveld, M. S. Klamkin (Canada), Vaclav Konecny (Czechoslovakia), M. Ram Murty and V. Kumar Murty (Canada), William Nuesslein, Paul Osterhus, T. K. Puttaswamy, J. M. Stark, and the proposer.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q631. We solve the more general problem $(xD^{n+1} - k^n xD + k^n n)y = 0$. The equation can be factored into $(D^n - k^n)(xD - n)y = 0$. Thus,

$$(xD-n)y = \sum_{i=0}^{n-1} A_i e^{k\omega^i x},$$

where ω is a primitive nth root of unity and the A_i's are arbitrary constants. Integrating again, we get

$$y = A_n x^n + x^n \sum_{i=0}^{n-1} A_i \int \frac{e^{k\omega^i x} dx}{x^{n+1}}.$$

Q632. A gallon, of course. Four replicas of the tetrahedron and one octahedron can be glued together to form a new tetrahedron with twice the linear dimensions, and hence eight times the volume of the small tetrahedra. They occupy half the volume of the new tetrahedron, which leaves the other half for the octahedron, which thus has four times the volume of each small tetrahedron. (One could also use the decomposition of an octahedron into eight tetrahedra and six octahedra.)

NEWS & LETTERS

OUR NEW LOOK

Regular readers of *Mathematics Magazine* will no doubt notice the changed format of this issue and may wonder as to the reasons for them. Some of the reasons were aesthetic, some economic. We adopted the larger page size of the *Monthly* for both reasons: in this format we can get more words per page (thereby publishing the same amount of mathematics in fewer pages), and at the same time reduce the crowded appearance of the former *Magazine* page. The cover illustration, together with the redesigned interior layout was motivated by our attempt to make the appearance as well as the content of the *Magazine* as attractive as possible.

Mathematics journals typically have a very long "pipeline" because of the extensive time it takes for a manuscript to move from editor to referee to editorial preparation to compositor, through galley and page proofs and finally into print. Dialog within the pages of the *Magazine* concerning published material is like communication with a distant galaxy, with response times measured in years.

The News and Letters section of the remodeled *Magazine* is designed to short-circuit this elaborate procedure. Here individuals who wish to comment on published articles may do so in the very next issue--provided they communicate with the editors shortly after receiving their copy of *Mathematics Magazine*. To insure that we use our very limited space as efficiently as possible, we reserve the right to edit letters to meet demands of available space.

We will use this space also for announcements and news of general interest, and invite readers to contribute appropriate items. The problems on p. 52 from the December 6, 1975 Putnam examination illustrate the short turnover time of the News and Letters section.

The Editors.

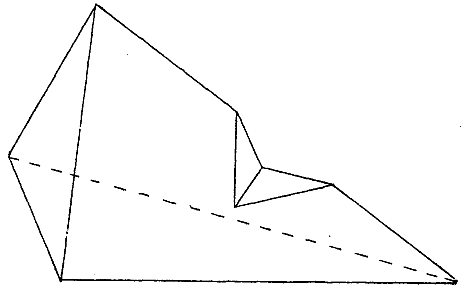
1976 CHAUVENET PRIZE

Professor Lawrence Zalcman of the University of Maryland was awarded the 1976 Chauvenet Prize for noteworthy exposition for his paper "Real proofs of complex theorems (and vice versa)" (*Amer. Math. Monthly* 81 (1974) 115-137). This prize, represented by a certificate and an award of $500, is the twenty-fourth award of the Chauvenet Prize since its institution by the Mathematical Association of America in 1925.

Lawrence Zalcman, a 1964 *summa cum laude* graduate of Dartmouth College, received his Ph.D. in 1968 at the Massachusetts Institute of Technology under the direction of Kenneth Hoffman. In addition to his research interests in complex analysis, he has a strong interest in the history and philosophy of mathematics and is currently occupied with problems which stress the intellectual (as opposed to the purely technical) content of mathematics.

POLYHEDRAL POSTSCRIPT

The result I proved on polyhedral face pairs (this *Magazine*, November 1975, pp. 289-90) had apparently been anticipated by Viktors Linis (this *Magazine*, September 1963) solving a problem (No. 508) proposed by David Silverman (this *Magazine*, January 1963). Linis' result is correct, but the proof is not: it implicitly assumed that a polyhedral face of n edges necessarily adjoins n distinct other faces. This is true for convex figures, but false



in general as is illustrated by the adjacent figure. Thus the methods of my paper are necessary to obtain the stated result. (I am indebted to Martin Gardner for pointing out this earlier reference.)

Benjamin L. Schwartz
Department of the Navy
Washington, D.C. 20370

WHAT'S AN EXPLICIT FORMULA?

In his article "An Explicit Formula for the kth Prime Number" (this *Magazine*, September 1975, pp. 230-232) Steven Regimbal in effect observes that $p_k < 2^k$ where p_k denotes the kth prime and then produces a summation of fewer than 2^k terms whose sum is p_k. But in fact the formula given for p_k is nothing more than $p_k = 0 + 0 + \ldots + p_k + 0 + \ldots + 0$. Certainly one cannot regard this as an explicit formula for the kth prime number.

Compare this with the following, where n is a perfect square. Define a function g on the set of positive integers by $g(m) = 0$ if $m^2 - n \neq 0$ and $g(m) = 1$ if $m^2 - n = 0$. Let k be any positive integer with $n < k^2$. Then

$$\sqrt{n} = \sum_{m=1}^{k} m \cdot g(m).$$

The equation is correct, but is this an explicit formula for $\sqrt{n}$? Obviously not.

James O. Chilaka
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ERRATUM

In our article "Seven Game Series in Sports" (this *Magazine*, September 1975, pp. 187-192) the equation on the first line of p. 189 should read $p^2 + q^2 = 1 - 2p(1-p)$. A p was dropped erroneously in the typing of the final manuscript.

Richard A. Groenveld
Glen Meeden
Iowa State University
Ames, Iowa 50010

THE BULGARIAN CONNECTION

B. Fisher's "A Fixed Point Theorem" (this *Magazine*, September 1975, pp. 223-225) is a known result. It appears, for example, as Theorem 4 in S.K. Chatterjea, "Fixed Point Theorems" (*C.R. de l'Acad. Bulgar des Sci.* 25 (1972) pp. 727-730). The contractive definitions of both Fisher and Kannan are special cases of a number of more general definitions. For a comprehensive treatment of this subject the reader may consult B.E. Rhoades, "A Comparison of Various Definitions of Contractive Mappings", to appear in *Trans. Amer. Math. Soc.*.

B.E. Rhoades
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PROOFS WITHOUT WORDS

I have the greatest respect for Mr. Rufas Isaacs and a profound admiration for the beautiful proof of the Pythagorean theorem which appears in the September 1975 *Mathematics Magazine*. But I am puzzled as to why it was printed there. The "proof without words" appears in *Scientific American*, October 1964, quoting an old Chinese manuscript of Chou Pei, and in *The World of Mathematics*, Volume I, by James R. Newman, quoting *The Great Mathematicians* by Turnbull. Surely both the editors and Mr. Isaacs are aware of these or other sources.

William H. Wertman
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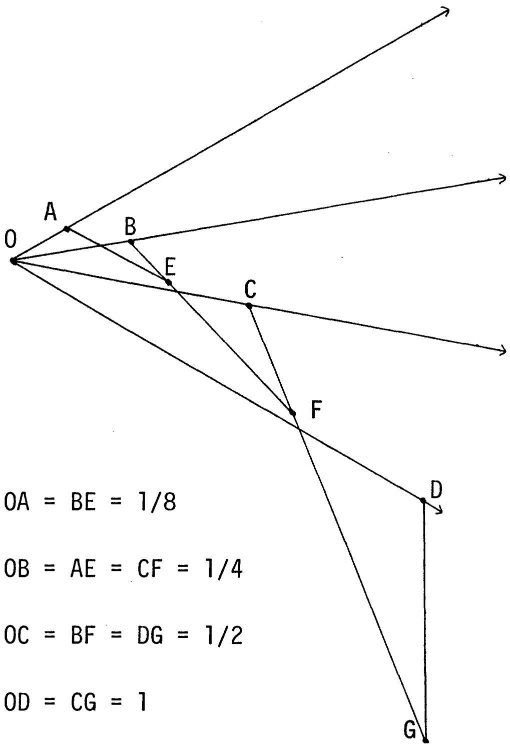
Maryland 20901

While the device presented in the September 1975 issue may trisect an angle, it is not the easiest to construct. Extensive cutting is required in the center pieces and, if made out of wood, the overlapping pieces will require bushings (especially those in the center). Below is a pattern for a device that I worked out several years ago (although I am sure it is not original) which trisects an angle, is easily constructed, and because of its basic design, is easily transformed into a device which will divide an angle into fourths, fifths, etc. This device, in addition to being expandable,



$OA = BE = 1/8$

$OB = AE = CF = 1/4$

$OC = BF = DG = 1/2$

$OD = CG = 1$

can be used on angles in excess of 360° and most importantly, can be built using a straight edge and a set of dividers since each key hole is a "multiple bisection."

George B. Miller
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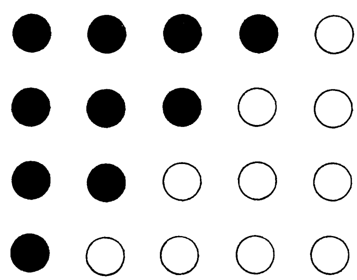
My little piece "Two Mathematical Papers Without Words", appearing in the September 1975 issue, has evoked several comments about the widespread past of the Pythagorean theorem proof therein. In addition to W.H. Wertman's observation, P.W. van der Pas informs me that it is the standard proof in Dutch highschools where it is often attributed to Multatuli, the pen name of author E.D. Dekker (1820-87). My colleague, E. Naddor, tells me a plastic model of the diagram is used in schools in Israel. In T.L. Heath's *Euclid's Elements*, Volume I, the proof seems attributed both to Bratschneider and Hankel.

I knew the proof was old but not that it was so ubiquitous. All I intended was to stress the rare and secluded pleasure of grasping a mathematical truth from visual evidence alone. Pedagogically, the purely visual spurs the young mind to its own thinking but has drawbacks for the teacher. Here is another example, a proof that $1 + 2 + 3 + \dots + n = n(n + 1)/2$.



One youngster may grasp the idea on his own; another may be baffled without some prompting. A third may see confirmation for the $n = 4$ depicted, but cannot perform the implicit induction which sees that there is nothing special about this case.

The trivial angle trisector is my idea, but, in view of age of the problem, it would be surprising were I not preceded.

Rufus Isaacs
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Editor's Note: We would like to encourage further contributions of proofs without words for the reasons mentioned by Rufus Isaacs and one other: we are looking for interesting visual material to illustrate the pages of the *Magazine* and to use as end-of-article fillers. What could be better for this purpose than a pleasing illustration that made an important mathematical point?

THE 1975 WILLIAM LOWELL PUTNAM MATHEMATICAL EXAMINATION

A-1. Supposing that an integer n is the sum of two triangular numbers, $n = \frac{1}{2}(a^2 + a) + \frac{1}{2}(b^2 + b)$, write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how x and y can be expressed in terms of a and b. Show that, conversely, if $4n + 1 = x^2 + y^2$, then n is the sum of two triangular numbers. [Of course, a, b, x, y are understood to be integers.]

A-2. For which ordered pairs of real numbers b, c do both roots of the quadratic equation $z^2 + bz + c = 0$ lie inside the unit disk $\{|z| < 1\}$ in the complex plane? Draw a reasonably accurate picture (i.e., 'graph') of the region in the real bc-plane for which the above condition holds. Identify precisely the boundary curves of this region.

A-3. Let a, b, c be constants with $0 < a < b < c$. At what points of the set $\{x^b + y^b + z^b = 1, x \geq 0, y \geq 0, z \geq 0\}$ in three-dimensional space R^3 does the function $f(x, y, z) = x^a + y^b + z^c$ assume its maximum and minimum values?

A-4. Let $n = 2m$, where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1 - \theta)^{-1}$ explicitly as a polynomial in θ,

$$a_k\theta^k + a_{k-1}\theta^{k-1} + \dots + a_1\theta + a_0,$$

with integer coefficients a_i.

A-5. On some interval I of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation $y'' = f(x)y$, where $f(x)$ is a continuous real-valued function. Suppose that $y_1(x) > 0$ and $y_2(x) > 0$ on I. Show that there exists a positive constant c such that, on I, the function $z(x) = c\sqrt{y_1(x)y_2(x)}$ satisfies the equation $z'' + z^{-3} = f(x)z$. State clearly the manner in which c depends on $y_1(x)$ and $y_2(x)$.

A-6. Let P_1, P_2, P_3 be the vertices of an acute-angled triangle situated in three-dimensional space. Show that it is always possible to locate two additional points P_4 and P_5 in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

B-1. In the additive group of ordered pairs of integers (m, n) [with addition defined componentwise: $(m, n) + (m', n') = (m+m', n+n')$] consider the subgroup H generated by the three elements $(3, 8)$, $(4, -1)$, $(5, 4)$. Then H has another set of generators of the form $(1, b)$, $(0, a)$ for some integers a, b with $a > 0$. Find a.

B-2. In three-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \dots$ of slabs of thicknesses $d_1, d_2, \dots$, respectively, such that Σd_i converges, prove that there is some point in the space which is not contained in any of the slabs.

B-3. Let $s_k(a_1, \dots, a_n)$ denote the k-th elementary symmetric function of $a_1, \dots, a_n$. With k held fixed, find the supremum (or least upper bound M_k of

$$s_k(a_1, \dots, a_n)/[s_1(a_1, \dots, a_n)]^k$$

for arbitrary $n \geq k$ and arbitrary n-tuples $a_1, \dots, a_n$ of positive real numbers.

B-4. Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that (i) B is topologically closed, and (ii) B contains exactly one point from each pair of diametrically opposite points on the circle?

B-5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \dots$. Show that $\Sigma f_n(1)/n! = e^e$.

B-6. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, then (a) $n(n + 1)^{1/n} < n + s_n$ for $n > 1$, and (b) $(n - 1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

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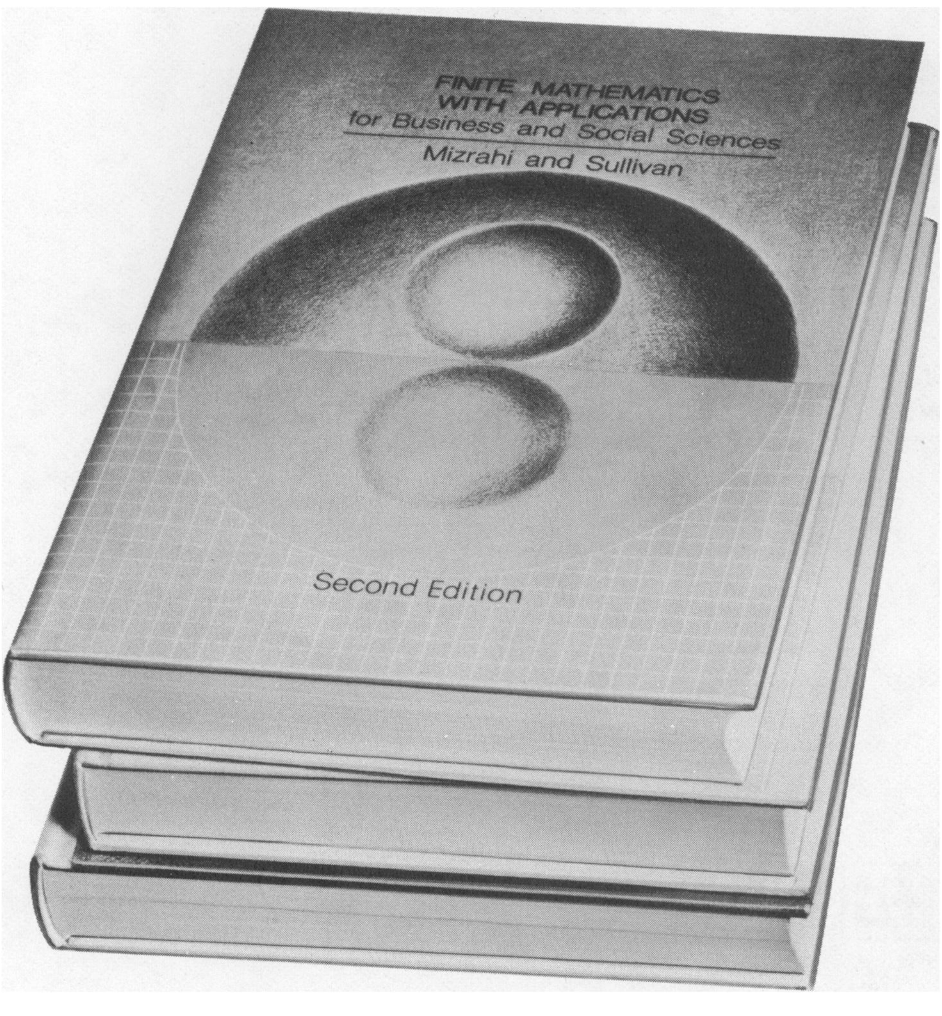

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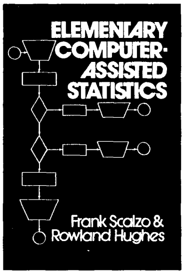

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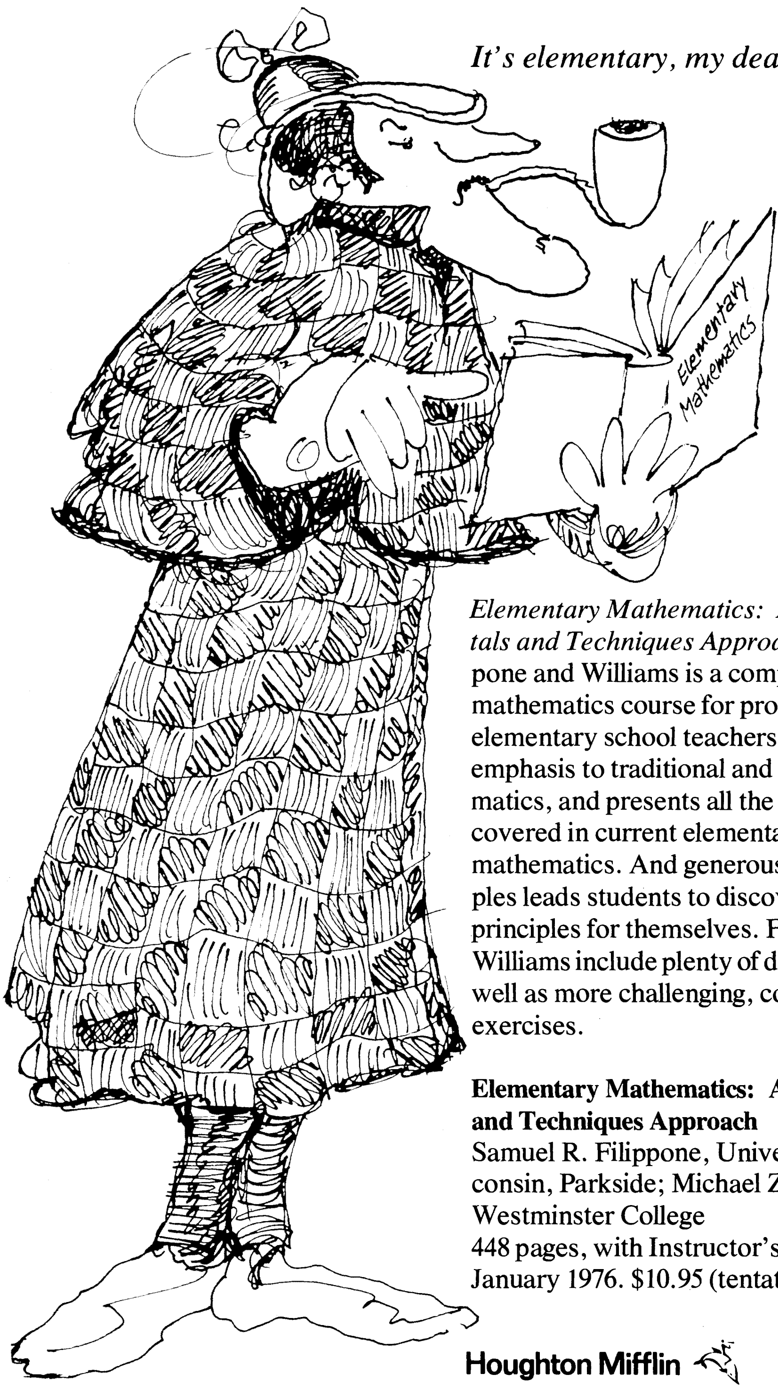

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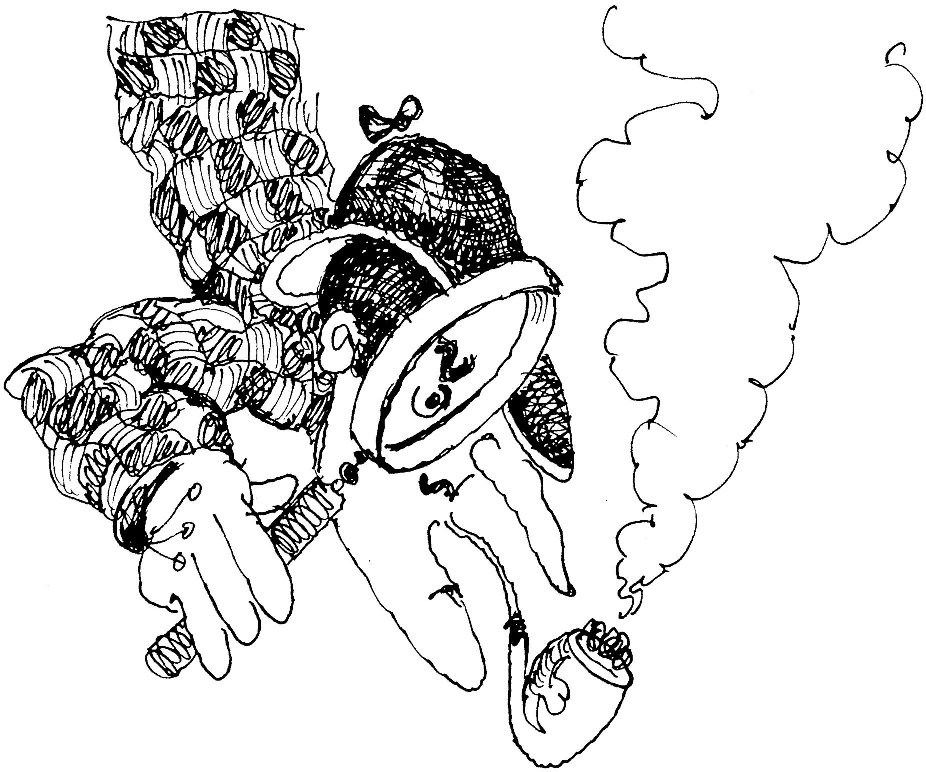

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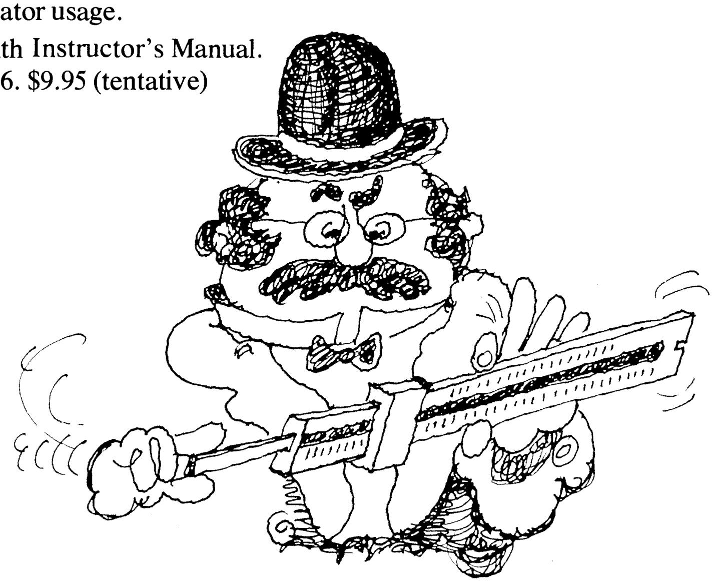

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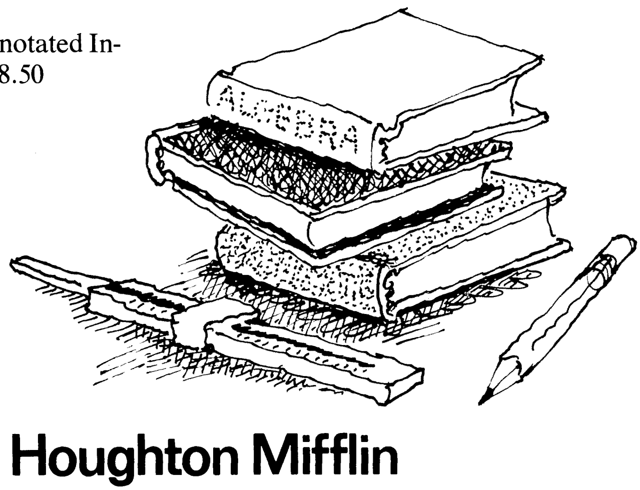

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